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RISK EVALUATION IN PRIVATE PENSION FUNDS THROUGH STOCHASTIC MULTI-STATE MODEL

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Introduction

During the last three decades, Italy has experienced several changes in the regulatory environment of the public pension system. These changes are due to the demographic, financial, and monetary changes that have progressively occurred during the last century which have gradually made the old system obsolete and financially unsustainable. The evolution of these variables has not been supported by an adequate adjustment of the pension system. For this purpose, from the 1990s a series of corrective measures were put in force in order to stop and resolve this imbalance. As a consequence, contributors have witnessed increasingly restrictive reforms. As is well known, this has led the government to regulate and encourage the use of supplementary pension instruments of a private nature.

The increase in the business volume of private pension funds, together with the strong growth in risk awareness brought by the recent periods of uncertainty in the financial markets, has led to a progressive demand for transparency and risk-based management of resources. To this end, the European regulation has tried to move towards the introduction of quantitative methods of risk assessment. However, these introductions have not received an excellent response from the pension sector due to the complexity and cost in terms of application and capital. Considering the recent events and the sensitivity of pension schemes to changes in demographic trends and market volatility, there is an increasing need for further protection of adherents.

This thesis provides an overview of the evolution of the Italian pension system with a focus on supplementary pensions. It also explores the risks affecting pension schemes and, capital needs to protect the pension fund's solvency. To do so, we have applied multi-state models to pension schemes, which allow the treatment of various retirement causes at the same time, such as disability, old age, and survivors' pensions. From this framework, combined with stochastic processes related to inflation and market return, a risk calculation model, based on numerical methods, has been developed.

The structure of the following paper is organized as follows. Chapter 1 offers a summary of the regulatory history that has defined the current structure of the Italian public and private pension systems. In this regard, we highlight the factors that led to regulatory changes, the increasing importance of complementary pension from the point of view of pension contributors and deepen the development of private sector's regulatory situation.
Chapter 2 reports concepts relating to the life actuarial technique and the relationships necessary for understanding the issues relating to pension schemes and multi-state models.
Chapter 3 reports theoretical results relating to Multiple decrements models, intended as a generalization of the classic life actuarial models in which death is considered as the only insured event, to this end we report a focus on what is defined as multi-state Markov models that we will then use in practical applications.
In Chapter 4 we describe the multi-state pension schemes we have created, Defined Benefit and Defined Contributions, we describe them and report the necessary assumptions, we report the analytical solutions necessary to calculate monetary movements, contribution rates, as well as technical provisions of the Defined Benefit scheme set up.
In Chapter 5 we discuss pension funds' solvency, we report the European Regulation Framework, consisting mainly of the *IORP I* directive and the following *IORP II* directive, as well as the risk measurement and disclosure tools proposed in the last decade by the European regulation in order to standardize disclosure to the market and the protection of members throughout Europe; the concept of risk measure and risk-based capital allocation is also described,

highlighting the stochastic processes relating to pension funds.

Finally, in Chapter 6, we discuss the theoretical and application notions relating to the main stochastic processes influencing a pension fund: longevity, inflation, and return on investment; we report the setup of the model necessary for the projection of cash flows and the related risk assessment; we carry out an analysis divided into three scenarios in order to highlight the impact that each individual stochastic process has on the funds, we then discuss the characteristics and compare the results among scenarios and type of funds.

1 The Italian Social Security System

In the last years the Italian social security system has been object of several changes, due to constant updates in the legislation. The overall Italian social security system is compounded by a *three pillars* structure, where the first pillar represents the public system while the second and the third represent the complementary pension system with respectively collective and individual adhesion. The public Italian social security system is a compulsory pay-as-you-go system, a structure such that the contributions paid by the workers in a certain calendar year are used to pay the contributions of the pensioners for the same calendar year.

It is clear how such a system is really linked with both the demographic structure of the adherents and employment status of the citizens in the country.

In the last years we have observed great changes in the demographic structure, in particular we can constantly observe an enlargement of the life expectancy, this main factor is one of the many which has driven the legislator to change the regulation concerning the social security system, both public and private.

The changes affecting the public system has often lead to a increasingly stringent rules that has naturally reduced the amount of benefits received by the pensioners together with an aggravation of age requirement to be entitled to receive the public pension.

Aside with this changes concerning the social security system we have observed an always larger importance of the complementary pension system which has become fundamental in order to maintain a sufficient level of benefits for future pensioners. In fact, in order to integrate the first pillar, private pension instruments were introduced and encouraged, those instruments permit to the worker to build its pension accordingly two sources: the public pension and a private component, which can be both mandatory or voluntary according with the different working categories.

1.1 The scheme of social security system

The social security system should be classified in order to better understand each component which will compound the future workers pension. The system is indeed financed by workers through the payment of contributions, which can have two different natures, they can be direct (paid specifically) or indirect (tax contributions). Typically the different nature of contribution match with two type of social security: *mandatory* or *voluntary*.

1.1.1 The three pillars of the social security

The mandatory system correspond mainly with the *Pillar I* and it concerns with the public system, where all the citizens are obliged to adhere to the state pension system, regardless if the citizen is a private employee, a public employee or a self-employed. The principle on which compulsory social security is based is to guarantee to all the retired workers a basic pension. The main institution of the Italian pension system is the *INPS*, to which, except a few self-employed workers or special categories, each worker must be registered.

Among the voluntary social security schemes we can further distinguish between two sectors: *collective basis* or *individual basis*. Thus, the *Pillar II* is based on voluntary participation in collective forms of pension funds, mainly linked to occupational status, and its aim is to integrate the benefits provided by the basic system and allowing an adequate standard of living in the retirement period.

For what concern the individual form of pensions, it can be associated with the *Pillar 3* of the social security system. In this case each person decides freely to join a pension fund or to stipulate an insurance contract for pension purposes. The main difference between collective pension funds and individual schemes consists in the fact that an adhesion of the first type involves a lower degree of freedom in terms of negotiations, this means that while with an individual pension scheme one could negotiate most part of the characteristics of its accumulation plan, in the second case typically are only offered preset agreements. On the other hand, collective pension funds generally allow to have advantages in terms of costs, thanks to standardized collective agreements for all members, and uniform investments.

1.1.2 The financial management systems

The financial management system regards the choice made ex-ante that will regulates the financial-actuarial system underlying the pension management and will basically link the contributions received by the institution to the benefits paid.

We can divide mainly in two kind of financial management systems: *Fully funded or capitalization system* and *Pay-as-you-go system*. In *Fully funded* the contribution requested to each worker are accounted in a mathematical reserve, those resources accumulated are invested by the collector and are then used in a later moment to pay for benefits when the insured become a pensioner. In capitalization we can distinguish between collective basis or individual basis according to the form of pension scheme chosen.

The *Pay-as-you-go system* instead works with a intergenerational agreement between workers and pensioners, in fact in this system the collected contributions are usually *not* set aside in reserves of the same subject who paid them but rather are used (on a collective basis) to pay pensioners benefits. We can further distinguish among *Pure pay-as-you-go* and *Partial pay-as-you-go*: in the first system the contribution collected on a certain period are directly used to pay the pension to the pensioners, the period of reference is typically from one to three years basis. The partial pay-as-you-go instead is a sort of mixed model, the contributions collected each reference period (for instance one year), are used to cover the expected present value of the future benefits estimated for only the *new pensioners* of that reference period.

It is clear how different management systems rely on different assumptions and are sensible to different indicators; for instance the Capitalization method is strictly sensible to the return on investment and also to the inflation rate variations. The Pure pay-as-you-go instead rely on two stable and favourable ratios: number of pensioners on number of workers and average pension on average salary.

1.1.3 Actuarial fairness

A further distinction concerns the methodology with which the actuarial balance is sought in order to calculate the contribution rates. We can differentiate between two way to setting up the system. The first way is the *Defined Benefit*, in this case the benefit received by the pensioner is fixed in advance, usually is used a percentage of the member's salary during the working period (*Replacement Rate*), then the necessary contributions are determined under specific assumptions made in order to discount future contributions and benefits; such assumptions can be: investment returns, salary dynamics, demographic mortality trends, etc. In this case we have to point out that the main risk is borne by the pension fund; in this framework the pricing is effectively made on some actuarial assumptions and so an unfavorable deviations of them in the long run would damage the fund.

In the *Defined Contribution* case instead the contribution rates that the worker pays are fixed in advance (typically a portion of the member salary), the benefit is then computed as *consequence* of both the rate of return on the market that the pension fund is able to obtain on the pool of contributions invested and the actual mortality rate of the insured population. In this kind of arrangement no guarantee is implied unless the underwriter provides for instance a capital protection or a death benefit.

1.1.4 Benefits

When we speak about social security system we think not only to the retirement benefits but also to other kind of benefits allowed by the welfare system; in particular there are main types of benefits provided by all the basics systems: Benefits in case of *retirement*, which includes both old-age retirement and early pension; Benefits in case of *full disability* or *partial disability allowance*, in this cases the benefits cover a total and permanent incapacity to work (full disability) or depends on the assessment of loss of work capacity of at least two-thirds; Benefit for *survivors*, in this case the survivors of a receiver of retirement/disability pension or survivors of a deceased worker continue to receive the pension benefits.
A further distinction concerns the calculation method of this benefits. We can differentiate between two methods to compute the first pension: *Contribution-based* calculation or *Salary-based* calculation. Contribution-based methodology is usually linked with capitalization management system, in fact according to this method, the first benefit is determined according to the contributions actually paid (or recognized) during the working life of the insured and appropriately capitalized at a rate of return. The total amount is then converted into a life annuity using actuarial coefficients depending on projected mortality and assuming future rates of return.
Instead according to the Salary-based methodology the first benefit is determined according to a function of the salaries defined ex-ante. The drawback of this method is that it has a limit of not always creating a correlation between the amounts paid by the worker and what is perceived as pension; indeed this methodology relies on some basics assumption depending on the definition of the salary function used to compute the benefits.

1.2 The evolution of Italian public social security system

The Italian pension system since its birth has been object of several legislative changes which can be divided in two temporal macro phases. The first phase starts with the building of the social security system and the further inclusion of several working categories, this part is characterized by an extension of pension benefits in terms of working categories and different forms of protection offered. These progressive changes has later lead to a constant worsening of public finance, which, together with a slow economy growth and a low birth rate has obliged the state to make some changes to reduce the amount of guarantees. The second phase indeed is characterized by several interventions in order to reduce the public expenditure through the reduction of pension benefits and the homogenization of the more privileged categories to the less expensive ones.

1.2.1 The birth of the mandatory social security system

The first prototype of pension system was introduced in the second half of 1800 for public sector employees and military, then we have in 1898, with the establishment of *National Fund of Social Security for Disability and Old Age of the workers* the first reference of pension system for private employees. Initially the pension scheme was on voluntary basis with as recipients only some particular categories of private workers and on a *Capitalization* structure.
In 1906 the *R&D* (retirement and disability) insurance for the covered categories were settled as mandatory. In 1919 with the law 603, the *R&D* insurance were extended to all private employees aged between 15 and 65 years old, with the exception of employees with remuneration exceeding *350 Lire* (Italian currency at that time) per month. The funded scheme though were not changed: the payroll taxes financed the system, those contributions were invested in

bonds or real estates and then sold in order to provide old age and disability benefits on a contributory basis.
After this firsts interventions, we have to wait until 30's in order to have some innovations. In 1933 during the fascist era, the National Fund of Social Security for Disability and Old Age of the workers was transformed in the current National Security Service: *INPS - Istituto Nazionale per la Previdenza Sociale*, in the meanwhile it was reduced the minimum age to be entitled to receive the pension, from 65 years old to 60 for men and 55 for women. In the following years, the state had difficulty of maintaining the financial balance during the Second World War; the high inflation that had affected the value and the real income of the pension reserves, led to the need for a revision of the system. Indeed the high inflation led to a reducing the value of benefits received by the pensioners and so a to an obligation by the state of high revaluation of those pensions.
This situation forced the passage from a fully funded scheme to a *Pay-as-you-Go* scheme. This mechanism, based on intergenerational solidality agreement, permitted to join about the positive and high ratio between active workers and pensioners, since at that time the life expectancy was shorter and the birth rate higher.
During the postwar period, with the establishment of the *Welfare State* model, we see an increasing intervention of the state and a progressive extension of the traditional assistance guarantees, indeed during the 50s and 60s, also due to the economic boom and the strong demographic increase has been applied several legislative intervention which increased the number of the working categories: direct growers, sharecroppers and settlers in 1957, artisans in 1959 and retailers in 1966, moreover we have to mention that in 1952 was introduced the minimum pension and in 1969 was introduced a social pension, guaranteed to all citizens over the age of 65 regardless of any previous contribution, as long

as they prove to be without sufficient income.
Among the regulatory interventions of those years it is necessary to mention the *Brodolini reform* (1969), it sanctioned the application of the pay-as-you-go system in a definitive way by defining the calculation method of the benefits (Salary based method) and by regulating the intervals of time used to compute the benefits depending on the category of workers. Furthermore it was introduced a system of revaluation of pension depending on the percentage variations of the index of consumer prices and the possibility to be entitled to receive the pension after a certain minimum year of contribution seniority (early age retirement).
The result obtained with this reform, in addition to subsequent regulatory interventions, was the configuration of a particularly onerous system that slowly aggravated public finances, especially during the 70's and 80's. In this specific period of time, there was a slowdown in economic expansion and demographic growth, which were fundamental prerequisites for the proper functioning of the pay-as-you-go system, effectively based on an intergenerational balance.

1.2.2 The 90s reforms

The state of the Italian pension system was characterized by a strong imbalance essentially due to the prospect of constantly increasing in social security benefits against the stationary nature of the contributions paid. The characteristics of this imbalance were made even stronger by the excessive inhomogeneity of the social security system, consisting of a variety of different managements and requirements among the several categories. The objectives of cost containment, performance reduction and system homogenization were no longer postponable over time and it was for this reason that in the 90s there were radical reversals in the regulatory field.
The first reform was the *Amato Reform*, constituted with two different de-

crees: the n. 503/1992 and n. 124/1993, the first was aimed to reform the social security system, while the second was aimed to a reorganization of the complementary pension scheme.
Focusing on the mandatory public system, the reform did not change the structure of the security system but rather it introduced some corrective measures aimed to: guarantee an higher match between contributions effectively paid and retirement salaries, contain the social expenditure and to curb the imbalance of the system.

Among the important innovations, the conditions for accessing the old-age pension were reformed. Specifically, the gradual raising of the age requirements, passing from 60 to 65 years old for men and from 55 to 60 for women, furthermore was introduced an extension of the minimum contribution period, passing from a minimum of 15 years to a minimum of 20 years. Stricter rules were also applied to the criteria for access to the early retirement pension, the minimum years of contribution seniority was settled to 35 for both private and public employees. For what concern the calculation of retirement benefits, the reform changed partially the function of the salary used to calculate it. In particular it increased the time span used to compute the *index salary*. The previous calculation criterion of first pension, (e.g for private employees) was based on the consideration only of the last 5 incomes before the retirement. The new regulation instead of 5, set to 10 or *all salaries* according to the seniority of contributions reached at the end of 1992. [1]
Finally, about the equalization of pensions, starting from 1994 it was foreseen to be annually made by an indexing to a consumer price index and not anymore to the minimum salary dynamic as was established some year before,

[1] These differentiation according to the seniority was introduced in order not to harm too much people who were closer to retirement.

moreover the revaluation was set to be annual and not anymore on six month basis.

Although the Amato reform represented a first rebuild of the Italian social security system, its provision were not enough relevant to overcome the unbalance and the inhomogeneity of the system in the early years of 90s; In particular despite the numerous corrections made by the pension system, INPS projections highlighted the inability of the social security system to guarantee an adequate financial balance.
The legislator intervened again with the law n. 335/1995, called *Dini Reform.* This reform responded to the needs for a structural revision of the social security system. While the acquired rights were maintained, the reform appointed the passage from a salary-based method to a *contribution-based* method to compute the retirement salary. The benefits were determined by capitalizing annually the contributions paid by workers at the average of nominal growth rate of GDP for the last 5 years and applying coefficients (called *Transformation coefficients*), representing the life expectancy at the time of accrual of the right to pension, to convert them into annuities. In particular, for employees it was recognized a contribution coefficient of 33%, a contribution coefficient of 20% was recognized to the self-employed category. At the same time, the contribution rate *actually* paid by the employees category was extended from 27% to 32%; though this increase did not entail any burden for the worker and employers since, at the same time, the contribution rate for family allowances and other management was reduced.
The transition to the contribution-based method for calculating retirement wages was, however, gradualized in the sense that a long transitional period was maintained within which the salary-based method continued to be valid. Therefore, three categories of workers were identified: the newly hired, to

whom the contribution-based calculation was directly applied; workers who at 31 December 1995 had matured less than 18 years of seniority, to whom a mixed system was adopted; and workers with more than 18 years of contributions, for which the salary-based methodology remained in force.
Although the strong innovative nature of the reform was recognized, the main criticism it received concerned the excessive timing required to fully implement it, with respect to the urgent need to contain social security expenditure.

1.2.3 The 2000s reforms

The years that followed were characterized by numerous interventions that aimed to speed up and rationalize the costs of the social security system and the impact that these had on GDP. Therefore, numerous legislative interventions were recorded, among which we will mention only the most significant.
The goal of rebalancing public finances, reducing the burden of social security costs as much as possible and shortening the transitory period provided for by Dini's provision, was the foundation on which the *Prodi reform* rested (law n. 449/1997). In particular the provisions of this reform concerned the acceleration some changes made by the previous reform, a further harmonization between private and public sector regarding the criteria of determination and equalization of benefits, and furthermore a revision of the contribution rates actually paid by certain working categories such the self-employed one (gradual growth from 15% to 19% within the next 15 years).
The law n. 243/2004, called also *Maroni reform*, introduced further innovations both in the field of public pension provision and in terms of complementary pensions. On the subject of compulsory social security, the reform has redesigned some of the characteristics, acting on two distinct fronts: first of all, the requirements for accessing the pension, both for old age and seniority, were raised starting from 1 January 2008; at the same time a mechanism that

would encourage the permanence on the job was introduced, better known as *superbonus*. Specifically, the requirements to access the early age retirement were gradually increased until 62 years of age in 2014 (with at least 35 years of contributions). This regulatory intervention was particularly discussed since it implied a sharp leap between the old regime requirements and the new ones. Also the requirements concerning the old age retirement were changed, in particular the men retirement age was settled at 65 years old, while the women one at 60 years old. As mentioned, the second front on which the reform intervened concerns the superbonus, which allowed the worker to receive a certain benefit if he voluntarily decided to temporarily renounce the retirement despite having achieved the requirements in terms of seniority; this tool aimed to reduce the people pensioning in the years 2007 and so was more intended to be an emergency measure than a long term solution. Another instrument designed by the Maroni reform, this time intended to reduce the long run expenditure, was the permission, for the women workers only, to retire, even after 2008, using the previous requirement (57 years old and 35 years of contribution seniority) but as long as they accepted that their pension would be fully calculated using contribution-based methodology.

The subsequent government, through the *Damiano reform* (law n. 247/2007), managed to solve the critical aspects of the previous reform, introduced some new tools and took care about the planning of the review of the transformation coefficients used to compute pension benefit through the contribution-based approach. In particular the reform abolished the sharp leap of three years (from 57 to 60) in the age requirement of seniority retirement introduced by Maroni; it instead partially revised the gradual increase in the requirements by introducing for the first time the so called *quota system*; this last is a system where the seniority pension can be obtained once is reached a certain quota, compounded by the sum between age and seniority. About the new tools

to contain the expenditure, the Damiano reform introduced also the *Window system* which provide a time lag between the period of maturity of the requirements and the starting date of the pension; for instance a worker that have reached its seniority requirement within the 31 of March, had to wait until July or October to receive the first pension according to its job category, July if employees, October if self-employed. Specifically they introduced four *windows* with the same time lag: End of March, June, September and December. Lastly the Decree also introduced the creation of a *Commission of experts* with the task of proposing by December 2008 changes to criteria for updating the transformation coefficients introduced in 1995 by Dini reform. In a later moment the commission identified some innovations in the criteria of computation of transformation coefficients; it indeed proposed that those coefficients should depend on macroeconomic, demographic and migratory dynamics, that they should include a solidality mechanism for lower pensions and should consider the relationship between the average life expectancy of the population and that of the individual sectors of activity. Furthermore it was envisaged that these coefficients will be reviewed on a three year basis.
For the next years we have to mention the law n. 102/2009 concerning anti-crisis measures and the law Decree n. 78/2010 regarding some urgent measures regarding financial and competitive stabilization. In particular the art. 22 ter of law n.102, following a sentence of the European Court, raises the minimum retirement age for old age workers of the public services; gradually bringing the the retirement age of female workers to 65 years old. Some other changes concerned the review of pension windows in order to contain the expenses, increasing the time gap to twelve months. Though, the main innovation made by this reform concern the introduction of an *automatic indexing mechanism* for the age of retirement. The requirement was indeed automatically linked to the life expectancy trend of the previous five years. The mechanism was

clarified by the law n. 78/2010 which specified that the automatic adjustment concerned both the minimum age and the minimum amount (calculated as the sum of the age and contribution years) required for seniority retirement. Beside was established that the life expectancy trend adjustment of the requirement would be done with specific decrees on a three year basis, according to the life expectancy trend calculated yearly by ISTAT.[2]

1.2.4 The Fornero reform and further changes

The most radical reform of the last years can be traced back to the much discussed *Monti-Fornero Reform* (law n. 214/2011), for which the seniority of contribution accrued after 31 December 2011 will be calculated according to the contribution-based model starting from 1 January 2012. In response to the European requests to eliminate all forms of discrimination between genders and inhomogeneity between categories of workers, the reform provides that by 2021, at the end of a transitional phase, homogeneous alignment will be achieved, where both men and women of the public or private sector, employed or self-employed, can receive the old-age pension after the age of 67 and with the possession of at least 20 years of contributions. Furthermore, with this provision, starting from January 1, 2012, the quota system has been abolished, to which the so-called early-pension has been replaced, which covers a substantially similar role but provides for a generally longer contribution period. In particular, we do not speak anymore of 40 years, but of 41 years and 3 months for women and 42 years and 3 months for men, in addition to the adjustment of the aforementioned requirements to the life expectancy index. Moreover, early-pension was provided with an additional requirement of minimum age, such that in the event of retirement at the age of less than 62 years, a percentage reduction is applied on the pension amount.[3]

[2] Italian central statistical Institute.

[3] Law n. 214/2011

In general, the new system appears highly rigid and penalizing for future retirees. At the same time, this reform process seemed necessary and fundamental to finally reach the resolution of the problem of financial imbalance that had been going on for several years. Therefore, became inevitable the use of complementary pension forms in order to achieve an adequate standard of living, leading to a growing demand for pension products to supplement the basic system.
Finally we have to mention one last change that has affects the pension system in the last years, the D.lgs 04/2019, also known as *Quota 100*. It introduced again a compound requirement to be pensionable, in particular it included between the start of 2019 and the end of 2021 an additional possibility of retirement, at age of 62 and a seniority of 38 years. Beside, the law also stopped the effect of life expectancy up to December 2026 on the early retirement criteria.

1.3 The Italian supplementary pension schemes

As highlighted in the previous paragraphs the issue of supplementary pensions in Italy has always sparked strong debates regarding the growing need to give it adequate regulation and to recognize the services as a fundamental pillar of the social security system.
Even if younger, like the public pension system, the supplementary pension has received during the last years many changes in terms of regulation and treatment. Apart from some minor changes, the main regulatory sources can be found contextually to some of the main reform of the public system, such as Amato reform, Dini reform and Maroni reform.

1.3.1 Establishment of the Italian supplementary pension system

The Amato reform, through Legislative Decree n. 124/1993 for the first time has taken steps to address the issues related to the supplementary pensions. Until that time, although without specific regulations, was operating only the voluntary forms of social security, mainly supplementary and offered mainly by the banking and insurance sectors. The starting point was represented by the need to complement the mandatory pay-as-you-go pension system with a supplementary type of pension, managed with a funded scheme; and this was made by offering a exhaustive regulation and by providing fiscal incentives to incentivize the worker to integrate their pension through the private supplementary pension system, in order to ensure higher levels of social security coverage. The Decree furthermore dictated detailed rules defining:

- *Recipients of social security*: for private and public employees of the different categories, for group of self-employed workers and freelancers,[4] for groups of worker members of production and labor cooperatives (introduced with the l. 335/1995) and for the people with the right to sign up to "Pension fund for people who carry out unpaid care work resulting from family responsibilities" (D.lgs. n. 565/1996).

- *The institutive sources of the forms of social security*: collective agreements, agreements, agreements among self-employed workers, or promoted by companies or trade unions.

- *Constitution and definition of control bodies of the fund and the persons authorized to manage the resources and to provide services*

[4] for self-employed and freelancers, also supplementary pension forms in a *defined benefit scheme* is actionable, aimed at ensuring a performance determined with reference to the level of income, or that of the compulsory pension treatment; for the remaining categories only a *defined contribution scheme* is available.

- *Fund management methods and annuities management*: Capitalization only, with the possibility to manage the annuities internally or by indirect management through agreements with other funds, insurance companies, SIM, banks, etc.
- *Tax aspects*
- *Regulation concerning pre-existing funds and definition of the new pension funds categories*: Closed or contractual pension funds, Open pension funds, Individual pension forms and the aforementioned pre-existing funds.
- *Rules concerning financing and benefits*
- *Introduction of a supervisory body (COVIP)*

About the different types of pension funds identified by the D.lgs. 124/1993, the legislator has defined four types, included the pre-existing funds: *Closed or contractual pension funds* are created through national or corporate collective agreement or agreements, they can be set up by individual company, by groups of company, by reference sector, professional association or orders or by territorial grouping, obviously the partecipation is also constrained in terms of reference sector or company, according to the internal rules;

Open pension funds are mainly set up by management companies, banks or insurance companies and represents an alternative to closed-end funds, they fulfill the dual role of individual and collective complementary forms, they does not have by definition any constraint in terms of reference category or territory.

Individual pension forms (introduced with the D.lgs. 47/2000) are a group of several individual forms that can be used to supplement the workers pension, typically are life insurance contracts, linked to products of class I or III such as Unit-linked products or with-profit contracts.

Pre-existing funds are a particular category of social institutions created before the 1993, for those entities, some exceptions are provided with respect to the constraints of the Amato reform.

For what concern instead the revenues, according to the Decree each fund can be founded by: contributions paid by the employers, contributions paid by the employees and finally by the termination indemnity (TFR). Instead, for what concern the benefits there is a strict regulation defining the modalities and the requirements to receive the pension. In particular the access to the services provided by the funds are defined by the Decree: a pensioner can access to the retirement salary for *old age* retirement, which requirements correspond to those of the public social security scheme plus an additional requirement which is at least five years of enrollment in the fund; or by *seniority* retirement, in this case the insured is entitled to receive the complementary pension whether he have reached at least 15 years of enrollment in the pension fund and an age of maximum 10 year less than the one required for the mandatory scheme.

It is also envisaged that in the event of termination of the requirements, the bylaws must provide for: transfer to another closed or open pension fund (as long as are passed at least three years from the enrollment and five years from the establishing of the fund), the surrender, and advances for healthcare costs/medical expenses or therapies and extraordinary interventions recognized by public facilities (to receive advances is by the way necessary an enrollment seniority of at least 8 years).

For what concern the payment modality, the law declared that the amount accrued, net of advances, upon reaching the requirement represents the benefit to which the member is entitled; the standard is to receive the benefit as an annuity, though the insured can decide to receive a portion (with a maximum of 50%) in lump sum. Only in the case that the 70% of the remaining part of accrued amount is lower than the 50% of the social allowance, the whole accrued benefit can be paid in lump sum.

As described, the Amato reform gave a strong regulation concerning the supplementary pension system; though despite the great expectations that saw supplementary pensions as a tool to overcome the sharp contractions in the public pension offer, it essentially found a weak adhesion and development of pension funds.

1.3.2 The review of the system: TFR and portability

A significant review of the supplementary pension system was introduced by D.lgs. n. 252/2005 which collected the objectives outlined in the law n. 243/2004 (Maroni reform). The effect introduced by the Decree were intended to be in force from the January 2008, though the Financial Act of 2006 revised the decree and anticipated its application in 2007. Overall, the legislative decree maintains some aspects of continuity with the previous regulatory provisions, among which the most significant concerns the principle of freedom to join the form of supplementary pension, for which the voluntary nature of this choice is reiterated. Despite these elements which guaranteed a certain consistency with what was previously established, the reform has brought substantial changes which we will analyze in detail below. First of all the decree expanded the abilities of the COVIP, whose power concern the guarantee of an adequate transparency, correctness of conduct, sound and prudent management of pension funds and, more generally, ensure the proper functioning of the social security system. In particular the powers of the COVIP consists in: authorization to the pension funds to exercise their profession and keeping the register of pensions fund authorized; approval of companies statutes, funds rules and verification of the adequacy of the organizational structure; assurance of proper management of funds, both in the accumulation phase and in the payment phase; definition of disclosure schemes in order to guarantee the transparency for the stakeholders.

A further innovative element is the introduction of the concept of *Portability*, with which a worker can transfer his position to another form of social security, obviously once the requirements designated by the regulation are met. According to this principle the insured has the freedom of choice (even more) the social security scheme to which allocate the financing items, TFR and employer payments, the latter within the limits set by collective agreements. Indeed the portability guarantee that after two years from the date of joining a supplementary pension scheme, the worker has the right to transfer the entire accrued pension position without the fund charging extra costs or limiting its possibilities. Finally, it is possible to note two further elements of novelty relating to Legislative Decree 252/2005: the tacit conferment of the TFR and the tax regime.

Before explaining the new treatment of TFR is necessary to focus on its concept and on its functionality. The termination indemnity (TFR) can be defined as an annual provision in the liabilities of companies of a part of the salaries that employees receive. In particular, to each employee is recognized the 6.91% of the annual salary. On each share, the worker accrues a nominal return of 1.5% plus 75% of the inflation rate each year. The amount that is gradually built up over time can be returned to the employee only at the time of retirement or in the event of termination of the employment relationship due to dismissal or change of business, or, again, as an advance to meet special needs . Advances can be requested: to support medical expenses, for which an amount not exceeding 75% of the amount set aside up to that point is paid, to purchase the first home, receiving a maximum amount of 75% only after they have expired eight years from registration, or for other needs, always after the same period

of time but obtaining a maximum value of 30%.
To incentivise the adhesions to pensions funds, the law allowed for the possibility of financing the future pension of each worker through its TFR. Before the 2005 a share of contribution to supplementary pension funds was already compound by a portion of the TFR, though the choice to join was not completely independent by the will of the company. The new mechanism of *silent-consent* has been introduced in order to permit to workers to confer automatically and directly the TFR in a form of supplementary pension. The mechanism works in a way such that the worker has to communicate within six months if they intend to continue to set aside the TFR (keeping it in the company) or to pay the fund. In the event no express choices is made by the employee within a six-month period, the TFR is automatically paid into an occupational pension fund. In case of tacit choice the TFR is automatically assigned to a collective pension form provided by collective agreements, including territorial ones. In the case of more than one applicable collective form are present, the TFR is destined to the one with the highest number of members. Again if the first two possibilities are not applicable, the TFR goes to the specific pension scheme set up by INPS, *FondINPS*.

2 Life Actuarial Notations

A pension fund, as well as an insurance life risk undertaker, must make many calculations inherent in human life in order to conduct its business. These calculations concern both the definition of the contributions to be paid and subsequently the calculation of the reserves necessary to meet one's commitments. In order to carry out these calculations typically the so-called *Life tables* and the actuarial mathematics behind are used.

2.1 Life table and its algebra

Life tables, taking into consideration a specific generation, describe the trend in the number of survivors from the moment of birth to death. A demographic table, starting from a theoretical population of 10,000 or 100,000 people living at birth, i.e. age zero, taking a specific sex as reference, reports for each subsequent age how many people remained alive and how many died until all the so called *cohort* is dead.[5] This definition is valid if the only cause for leaving the table is death. In fact actuarial tables can be created with other causes of exit such as disability in order to price also other kind of benefit instead of only death; in this case we speak about *Multiple decrement tables*, but we will return on this later. Life tables described above are thus one of the two main types of single decrement life tables, those are called *Cohort life table* since they show the probability of death of people with the same year of birth, over the course of their lifetime. Another kind of life tables indeed are the *Period life tables*, those show instead the current probability of death for people of different ages in the current year.

The demographic tables are typically compiled according to the *biometric functions*, which are a function of age x and allow to determine the probability of

[5] The terminal age of the cohort is tipically represented by ω and it is usually set up around 110 to 120 years old.

existence in life after t years, or the probability of death within that age in a homogeneous group of individuals. These functions are l_x and d_x where the first indicates the number of people who have reached age x with x between 0 and ω, while the second represents the people dead at age x, such as:

$$d_x = l_x - l_{x+1} \tag{1}$$

From this we can deduce that l_x decreases over time until it reach 0 at $x = \omega$. While d_x usually increases as x increases as the more extreme ages are characterized by higher *mortality rates*.
The demographic tables can further be distinguished, according to the characteristics they present, in population tables and market tables. The former are drawn up on the basis of information regarding the entire population of a country. In the case of Italy, these are for instance drawn up by the ISTAT, they are distinguished by sex and are usually called SIM and SIF which respectively indicates the male and female tables. The latter are obtained using data relating to the observations relating to the insured persons with an insurance company or, as in the case in question, the members of a pension fund. Among these we find, for example, the IPS55 and the A62, based on mortality for registered persons of the generation 1955 and 1962. Moreover we should remark that the latter differ from SIM/SIF not only for the fact that they are selected tables but also because they are projected.

2.1.1 Probability of life and death for single individuals

The frequency deduced from a survival table with which an individual of initial age x is still alive at age $x + t$ is taken as a measure of the *probability of life*:

$${}_tp_x = \frac{\ell_{x+t}}{\ell_x} \tag{2}$$

In order that an individual of age x is alive after $t + n$ periods it takes that he reaches the age $x + t$ and, having reached this, is still alive after n years, therefore by the theorem of probability decomposition:

$$ {}_{t+n}p_x = {}_tp_x \cdot {}_np_{x+t} \tag{3} $$

As n increases, the probability ${}_np_x$ tends towards 0, or rather, defining as the extreme age ω, we have that: ${}_{\omega-x}p_x = 0$.

Considering instead unit intervals it is possible to define the *annual survival rate* ${}_1p_x = p_x = \frac{\ell_{x+1}}{\ell_x}$, and subsequently through intervals of one year the probability of surviving t years can be expressed as a function of the annual probability of survival:

$$ {}_tp_x = \prod_{s=1}^{t} p_{x+s} \tag{4} $$

If we consider an individual of age x, the probability that he does not reach age $x+t$, i.e. that death occurs within t years, is:

$$ {}_tq_x = 1 - {}_tp_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} \tag{5} $$

As seen before, in case of $t = 1$ we have the annual mortality rate:

$$ q_x = 1 - p_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} = \frac{d_x}{\ell_x} \tag{6} $$

which is actually equal to the ratio between the number of people alive at age x and the number of death related to the same year.

The sequence of $q_0, q_1, q_2, ..., q_x, q_{x+1}, ..., q_{\omega-1}$ can be identified as *mortality table*.

The probability that a subject die in between the age of x and $x+t$ such as $x, x+1, x+2, ..., x+t$ are independent by the others and are defined as:

$$ q_x, {}_{1/1}\, q_x, \cdots, {}_{t-1/1}\, q_x \tag{7} $$

So the probability that the death occurs before than t years can be also expressed as follow:

$$
{}_tq_x = q_x + {}_{1/1}q_x + \ldots + {}_{t-1/1}q_x = \frac{1}{\ell}\sum_{x=0}^{t-1} d_{x+s} \tag{8}
$$

According to the *compound probabilities* it is also possible to express the mortality rates in function of life probabilities:

$$
{}_{m/n}q_x = {}_mp_x \cdot {}_nq_{x+m} = {}_mp_x\,(1 - {}_np_{x+m}) \tag{9}
$$

2.1.2 Probability of survivorship for multiple lives

This theory can be extended to the case of multiple lives. A typical application of this extension is commonly found in pension plans and is the *joint-and-survivor annuity option.*

To compute probabilities or actuarial present values associated with the survival of multiple lives, the joint distribution of the future lifetime random variable must be available. So the time-until-failure of a specific status depends on both the future lifetimes of the lives involved and the dependence between them. Typically an assumption of independence among future lifetimes has traditionally been made. Under this assumption, only marginal distributions are needed.

When facing with multiple lives, different cases can be defined:

- *joint-life status*, considering m members with age $x_1, x_2, ..., x_m$ and under independence, we have that the survival probability is:

$$
{}_tp_{x_1,x_2,\ldots,x_m} = \prod_{h=1}^{m} {}_tp_{x_h} \tag{10}
$$

  this case is important for instance to price a contract which survive as long as all the members of a set of lives survive and fails upon the first death occurs;

- *last-survivor status*, in case two members aged x_1 and x_2 respectively, the probability that at least one is alive in t years is defined as:

$$ {}_tp_{\overline{x_1,x_2}} = {}_tp_{x_1} + {}_tp_{x_2} - {}_tp_{x_1,x_2} \tag{11} $$

this probabilities concern a contract which survives as long as at least one member of a set of lives is alive and so fail upon the last death.

2.2 Random Lifetime

Another way to treat the lifetime and the relatives probability is by using a stochastic definition of the *time-to-death*. The remaining lifetime of an individual aged x is clearly a random variable, that we will define with T_x. It follows that the age of the individual at death is a random variable too and is given by $T_x + x$. The possible outcome of T_x are the positive real numbers and typically is usual to set $\omega - x$ as the maximum possible outcome. Given T_0, which represents the total lifetime of an individual of age 0, namely a newborn, the remaining lifetime of an individual of age x is:

$$ T_x = T_0 - x \mid T_0 > x \tag{12} $$

When a life table is available thus the following probabilities can be immediately derived:

$$ \begin{aligned} P\left(T_x > t\right) &= {}_tp_x = \tfrac{l_{x+t}}{l_x} \\ P\left(T_x \leq t\right) &= 1 - {}_tp_x = {}_tq_x = \tfrac{l_x - l_{x+t}}{l_x} \\ P\left(h < T_x \leq t+k\right) &= {}_{t/k}q_x = \tfrac{l_{x+t} - l_{x+t+k}}{l_x} \end{aligned} \tag{13} $$

2.2.1 Survival function and curve of death

Let $f_0(x)$ and $F_0(x)$ denote, respectively, the probability density function and the distribution function of T_0, while we denote with $S_0(x) = 1 - F_0(x)$ the *Survival function*. In particular, $F_0(x)$ expresses, by definition, the probability of a newborn dying within x years. Thus, according to the usual notation:

$$F_0(x) = P[T_0 \leq x] = {}_xq_0 \tag{14}$$

Since the following relation holds between $f_0(x)$ and $F_0(x)$:

$$F_0(x) = \int_0^x f_0(t)dt \tag{15}$$

and, as usually, assuming that for $x > 0$ the pdf $f_0(x)$ is a continuous function. Then we have:

$$f_0(x) = \frac{dF_0(x)}{dx} = -\frac{dS(x)}{dx} \tag{16}$$

Moving to the remaining lifetime at age x, and so with $T_x|x > 0$, the following relations link the distribution function and the pdf of T_x with the analogous functions relating to T_0:

$$F_x(t) = P\left[T_x \leq t\right] = \frac{P\left[x < T_0 \leq x + t\right]}{P\left[T_{0>x}\right]} = \frac{F_0(x+t) - F_0(x)}{S(x)} = {}_tq_x \tag{17}$$

and so also:

$$P\left[T_x \geq t\right] = \frac{P[T_0 > x + t]}{P\left[T_{0>x}\right]} = \frac{S_0(x+t)}{S_0(x)} = {}_tp_x \tag{18}$$

2.2.2 Mortality intensity

The *Mortality intensity* or *Force of mortality* provides a tool for a fundamental statement assumptions about the behavior of an individual mortality as function of the attained age. Given that:

$$
\begin{aligned}
{}_{\Delta x}q_x &= F_x(\Delta x) = P\left(T_x \leq \Delta x\right) = P\left(T_0 \leq x + \Delta x \mid T_0 > x\right) = \\
&= \frac{P\left(x < T_0 \leq x + \Delta x\right)}{P\left(T_0 > x\right)} = \frac{S(x) - S(x + \Delta x)}{S(x)}
\end{aligned} \tag{19}
$$

the Force of mortality is:

$$
\mu_x = \lim_{\Delta x \to 0} \frac{{}_{\Delta x}q_x}{\Delta x} = \lim_{\Delta x \to 0} \frac{S(x) - S(x + \Delta x)}{\Delta x S(x)} = -\frac{S'(x)}{S(x)} = \frac{f_0(x)}{S(x)} \tag{20}
$$

and so considering the boundary condition $S(0) = 1$ we have that:

$$
\begin{aligned}
\int_0^x \mu_u du = \int_0^x -\frac{S'(u)}{S(u)} du = -\int_0^x \frac{d \ln[S(u)]}{du} du = \\
- \ln[S(u)]|_0^x = = \ln[S(0)] - \ln[S(x)] = -\ln[S(x)]
\end{aligned} \tag{21}
$$

and so:

$$
S(x) = e^{-\int_0^x \mu(u) du} \tag{22}
$$

In this way, all survival probabilities and mortality rates can be derived, hence:

$$
{}_tp_x = \frac{S(x+t)}{S(x)} = \frac{e^{-\int_0^{x+t} \mu(u) du}}{e^{-\int_0^x \mu(u) du}} = e^{-\int_x^{x+t} \mu(u) du} \tag{23}
$$

3 Multiple Decrement Models

Calculation of premiums, for a variety of insurance products, is one of the important computations in insurance business. In the simplest setup, the interest rate is assumed to be constant during the lifetime of the policy, and the amount payable to individual is relative on the event of death. Though, as we have seen above the benefit may be payable to a group, and then it is necessary to define when to pay the benefit. Depending on this last we can use the approach of join life status or last survivor status. In both the situations, the future life time is the only underlying random variable, the uncertainty lies in *when* the death of the subjects will occur.

Survivorship models incorporating two random mechanisms, *time to termination*, and *various modes of termination* are known as *multiple decrement models*. Through these models we move from the classical setup of life insurance mathematics, and we move through a more complex analysis involving different benefits payable according with the reason of termination.

Multiple decrement models are also commonly in industrial application and in public heath insurance; for instance can be interesting to study the incidence rates for various diseases, so data are collected on the cause of death, along with the data on age at death. Though a major application of this models is actually in pension plans. In pension schemes. For instance the different causes of termination could be withdrawal, disability, death or retirement. The benefit paid in fact usually depends upon the cause of termination as well as the contribution seniority. As a consequence, the actuarial present value of the benefits depends on the specific insured event along with the future life time of an individual.

According to this framework the theory of life table when there is a single mode of exit can be extended to a more general theory of multiple decrement models involving the effect of several causes of decrement on a group of individuals.

3.1 The random variables of multiple decrement models

We define now again the continuous random variable $T_x = T$ extending further its concept by defining it such as the *time to termination* from a status in a generic decrement model. In the previous chapter the variable Tx was defining the *life expectancy* random variable, which can be seen also as *time-to-death*, according to its definition so it was just a specific single decrement way to define the *time to termination.* Though, since there may be more than one cause for termination from a given status, we define $J_x = J$ with values $1, 2, 3, ..., m$ a discrete random variable representing the cause of decrement.
We define with $h_x(j) = h(j) = P[J(x) = j], \quad j = 1, 2, \dots, m$ the probability density function of J and with $g(t)$ the probability mass function of T. Since T is a continuous random variable, while J is a discrete random variable we cannot compute the joint probability density function or joint probability mass function of these two random variables. So we define ex-ante the joint distribution function in order to find the joint probability of interest.
Suppose that the probability of decrement in $(t, t+dt)$ due to the specific cause j is:

$$f(t, j)dt = P[t < T \leq t + dt, J = j] \tag{24}$$

Then the probability of decrement in the interval $[a, b]$ due to cause j and the probability of decrement in the interval $[a, b]$ due to *any cause* are respectively given by:

$$P[a \leq T \leq b, J = j] = \int_a^b f(s, j)ds \quad \text{and} \quad P[a \leq T \leq b] = \sum_{j=1}^{m} \int_a^b f(s, j)ds \tag{25}$$

where the probability mass function $h(j)$ and the probability density function

$g(t)$ are related to $f(t, j)$ as follows:

$$g(t) = \sum_{j=1}^{m} f(t, j) \tag{26}$$

and,

$$h(j) = P[J = j] = \int_0^\infty f(s, j) ds. \tag{27}$$

The *Actuarial notation* for the functions of the time-until-decrement random variable and cause of termination in the multiple decrement model are as follows: ${}_tq_x^{(j)}$ denotes the probability of decrement in $(x, x + t)$ due to cause j and is given by:

$${}_tq_x^{(j)} = P[T(x) \leq t, J(x) = j] = \int_0^t f(s, j) ds. \tag{28}$$

Let's define with the symbol (τ) the total probability of decrement, which refers so to all causes, then remarking the results found in the section 2.2, we have:

$${}_tq_x^{(\tau)} = P[T \leq t] = G(t) = \int_0^t g(s) ds \tag{29}$$

$${}_tp_x^{(\tau)} = 1 - {}_tq_x^{(\tau)} = P[T > t] = \frac{S(x + t)}{S(x)} = e^{-\int_0^t \mu_{x+s}^{(\tau)} ds}, \tag{30}$$

$$\mu_x^{(\tau)}(t) = \frac{g(t)}{1 - G(t)} = -\frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tp_x^{(\tau)} = \frac{\mu_{x+t}^{(\tau)} t p_x^{(\tau)}}{{}_tp_x^{(\tau)}} = \mu_{x+t}^{(\tau)} \tag{31}$$

where ${}_tq_x^{(\tau)}$ denotes the distribution function of T_x at t, it is chance of decrement of (x) in $(x, x + t)$ conditional on survival up to age x;

${}_tp_x^{(\tau)}$ denotes the survival function of T_x at t, which has seen above is the probability of survival of x up to $x + t$;

while $\mu_{x+t}^{(\tau)}$ denotes the force of decrement corresponding to the *life expectancy*

random variable at age $x+s$, where decrement can occur due to any one of the m causes. $\mu_x^{(\tau)}(t)$ instead denotes the force of decrement corresponding to the random variable $T(x)$ at t. As in single decrement model, the force of decrement of $T(x)$ at t is the same as the force of decrement of the random variable *life-length* at x.
The force of decrement due to cause j , conditional on survival of (x) to $x+t$ is defined as:

$$\mu_x^{(j)}(t) = \frac{f(t,j)}{1-G(t)} = \frac{f(t,j)}{tp_x^{(\tau)}} \tag{32}$$

Since $f(t,j)$ does not have interpretation of joint density, $\mu_x^{(j)}(t)$ also does not have x interpretation of conditional joint density. But $f(t,j)dt$ with $j = 1, ..., m$ and $t \geq 0$, can be expressed as:

$$\begin{aligned} f(t,j)dt &= P[t < T(x) \leq t + dt, J = j] \\ &= P[T > t]P[t < T(x) \leq t + dt, J = j \mid T > t] \\ &= {}_tp_x^{(\tau)} \mu_{x+t}^{(j)} dt \end{aligned} \tag{33}$$

By substituting the last results in the equation 32, we get that $\mu_{x+t}^{(j)} = \mu_x^{(j)}(t)$.
Intuitively it is clear that *the total force of decrement is the addition of the individual forces*, this can be proved by some simple passages:

$${}_tq_x^{(\tau)} = \int_0^t g(s)ds = \int_0^t \sum_{j=1}^m f(s,j)ds = \sum_{j=1}^m \int_0^t f(s,j)ds = \sum_{j=1}^m {}_tq_x^{(j)} \tag{34}$$

and by applying the derivative to both sided:

$${}_tq_x^{(j)} = \int_0^t f(s,j)ds \Rightarrow \frac{d}{dt} {}_tq_x^{(j)} = f(t,j) \tag{35}$$

we have that:

$$
\begin{aligned}
\mu_x^{(\tau)}(t) &= \frac{g(t)}{1-G(t)} = \frac{1}{{}_tp_x^{(\tau)}}\frac{d}{dt}{}_tq_x^{(\tau)} = \frac{1}{{}_tp_x^{(\tau)}}\frac{d}{dt}\sum_{j=1}^{m}{}_tq_x^{(j)} \\
&= \frac{1}{{}_tp_x^{(\tau)}}\sum_{j=1}^{m} f(t,j) = \sum_{j=1}^{m}\frac{f(t,j)}{{}_tp_x^{(\tau)}} = \sum_{j=1}^{m}\mu_x^{(j)}(t).
\end{aligned}
\tag{36}
$$

The conditional probability mass function of J given decrement at time t is given by:

$$
h(j \mid T=t) = \frac{f(t,j)}{g(t)} = \frac{{}_tp_x^{(\tau)}\mu_{x+t}^{(j)}}{{}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)}} = \frac{\mu_{x+t}^{(j)}}{\mu_{x+t}^{(\tau)}} = \mu_{x+t}^{(j)} / \sum_{j=1}^{m}\mu_{x+t}^{(j)} \tag{37}
$$

With the definitions of $\mu_x^{(j)}(t)$ and $\mu_{x+t}^{(j)}$, we have for $j = 1, 2, 3, ..., m$,

$$
f(t,j) = {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}, \quad h(j) = {}_\infty q_x^{(j)}, \quad g(t) = \sum_{j=1}^{m} f(t,j) = {}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)}
$$

and

$$
{}_tq_x^{(j)} = \int_0^t f(s,j)ds = \int_0^t {}_sp_x^{(\tau)}\mu_{x+s}^{(j)}ds
$$

Therefore, the probability ${}_tq_x^{(j)}$ of decrement between the ages x to $x+t$ due to cause j depends on ${}_sp_x^{(\tau)}$, $0 \leq s < t$, and so on all the component forces of decrement.

Hence, when the forces for decrements other than j are increased,

$$
{}_tp_x^{(\tau)} = 1 - {}_tq_x^{(\tau)} = 1 - \sum_{j=1}^{m}{}_tq_x^{(j)} \tag{38}
$$

is reduced, and hence ${}_tq_x^{(j)}$ also gets reduced.[6]

[6] In view of this phenomenon, multiple decrement theory is also known as the *theory of competing risks* in survival analysis.

3.2 Multiple decrement tables

Multiple decrement table (MDT) is an extension of a single decrement table. In this setup the column of d_x is divided in m columns corresponding to the m causes of decrement.

In order to understand how is builded a Multiple decrement table let's suppose that we have a group of $l_a^{(\tau)}$ lives, each of age a years. We assume that each person's life has the *same* joint distribution of time-to-decrement and cause of decrement, which is specified by the joint probability seen in the eq. 33. Assume that $L_x^{(\tau)}$ is a random variable indicating the number of survivors at age x out of the $l_a^{(\tau)}$ lives in the original group at age a. Then $L_x^{(\tau)}$ can be expressed as:

$$L_x^{(\tau)} = \sum_{i=1}^{l_a^{(\tau)}} Z_i \tag{39}$$

where Z_i is defined as $Z_i = 1$ if the ith life survives up to age x, $x \geq a$, and 0 otherwise. Then $E(Z_i) = P[Z_i = 1] = P[T(a) \geq x]$, the same for all i. Thus, expectation of $L_x^{(\tau)}$, denoted by $l_x^{(\tau)}$, is given by:

$$l_x^{(\tau)} = E\left(L_x^{(\tau)}\right) = l_a^{(\tau)} P[T(a) \geq x] = l_a^{(\tau)}\,{}_{x-a}p_a^{(\tau)} \tag{40}$$

So, as in single decrement table, we get:

$$l_{x+1}^{(\tau)} = l_a^{(\tau)}\,{}_{x+1-a}p_a^{(\tau)} = l_a^{(\tau)}\,{}_{x-a}p_a^{(\tau)} p_x^{(\tau)} = l_x^{(\tau)} p_x^{(\tau)} \tag{41}$$

To obtain the analogue of ${}_td_x$, for each such life, a *Bernoulli random variable* Y_j is defined as $Y_j = 1$ if an individual from original group of $l_a^{(\tau)}$ individuals suffers decrements in $(x, x+n)$, $x \geq a$, due to cause j, and 0 otherwise. Then:

$$
\begin{aligned}
& P\left[Y_j=1\right] \\
& =P[x-a \leq T(a) \leq x+n-a, J(a)=j] \\
& =\int_{x-a}^{x+n-a} {}_t p_a^{(\tau)} \mu_{a+t}^{(j)} dt \\
& =\int_0^n {}_{u+(x-a)} p_a^{(\tau)} \mu_{u+x}^{(j)} du, \quad \text{ with } t-(x-a)=u \\
& =\int_0^n {}_{x-a} p_a^{(\tau)} {}_u p_x^{(\tau)} \mu_{u+x}^{(j)} du \\
& ={}_{x-a} p_a^{(\tau)} \int_0^n {}_u p_x^{(\tau)} \mu_{u+x}^{(j)} du \\
& ={}_{x-a} p_a^{(\tau)} {}_n q_x^{(j)}
\end{aligned} \tag{42}
$$

Suppose that the random variable ${}_n D_x^j$ denotes the number of lives who leave the group between ages x and $x+n$, with $x \geq a$, due to the cause j. Then:

$$
{}_n D_x^{(j)}=\sum_{i=1}^{l_a^{(\tau)}} Y_{ji} \tag{43}
$$

Its expected value is denoted by ${}_n d_x^j$ and is given by:

$$
{}_n d_x^{(j)}=E\left({}_n D_x^{(j)}\right)=l_a^{(\tau)} P\left[Y_j=1\right]=l_a^{(\tau)} {}_{x-a} p_a^{(\tau)} {}_n q_x^{(j)}=l_x^{(\tau)} {}_n q_x^{(j)} \tag{44}
$$

When $n=1$ we have so: $d_x^{(j)}=l_x^{(\tau)} q_x^{(j)}$.

If we consider instead the number of decrements due to all the causes between $x, x+n$, with $x \geq a$, we have:

$$
{}_n D_x^{(\tau)}=\sum_{j=1}^m {}_n D_x^{(j)} \tag{45}
$$

Taking the expectation, we get:

$$
{}_n d_x^{(\tau)}=E\left({}_n \mathcal{D}_x^{(\tau)}\right)=\sum_{j=1}^m {}_n d_x^{(j)}=\sum_{j=1}^m l_x^{(\tau)} {}_n q_x^{(j)}=l_x^{(\tau)} {}_n q_x^{(\tau)} \tag{46}
$$

and obviously, with $n=1$: $d_x^{(\tau)}=l_x^{(\tau)} q_x^{(\tau)}$. Finally we denote:

$$l_{x+1}^{(\tau)} = l_x^{(\tau)} p_x^{(\tau)} = l_x^{(\tau)} \left[1 - \sum_{j=1}^{m} q_x^{(j)}\right] = l_x^{(\tau)} - \sum_{j=1}^{m} d_x^{(j)} = l_x^{(\tau)} - d_x^{(\tau)} \tag{47}$$

These results enable us to obtain $l_x^{(\tau)}$ and $d_x^{(j)}$ values from $p_x^{(\tau)}$ and $q_x^{(j)}$ values. The table showing the values of $p_x^{(\tau)}$ and $q_x^{(j)}, j = 1, \ldots, m$, or $l_x^{(\tau)}$ and $d_x^{(j)}$ $j = 1, \ldots, m$, and for integral values of x, is known as a *multiple decrement table.*

Age x	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$
25	100000.00	176.74	12149.68	467.30	0.00
26	87206.41	163.44	10594.72	407.49	0.00
27	76040.88	151.41	9237.66	355.29	0.00
28	66296.61	140.51	8053.37	309.74	0.00
29	57793.08	130.60	7019.89	270.00	0.00
30	50372.66	125.27	3392.85	242.35	0.00
...	...	...	...	...	...
59	6664.37	175.67	0.00	31.83	381.99
60	6074.89	174.89	0.00	28.98	347.76
61	5523.25	173.68	0.00	26.31	315.75
62	5007.51	172.00	0.00	23.82	285.84
63	4525.86	169.80	0.00	21.49	257.92
64	4076.65	167.07	0.00	19.33	231.90

Table 1: Above an insurance multiple decrement table, where the decrements are due to: death, witihdrawal, disablity and retirement.

3.3 Multi-state Transition Models

Multiple decrement model describes the probabilities of transition from state 0 to state j at various time points. In this setup, transitions from j to 0 or

transitions between any two states i and $j, i \neq j = 1, 2, ..., m$, are not possible. Often though, in some context like the ours it is necessary a framework which is able to describe also passages among the m states.
An example could be health studies, where it is surely important to understand the passages from healthy to sick, but is also fundamental to study the further development of a specific illness (identifiable by a specific state), which is necessarly related to a further state *death*. For pension funds indeed, it is necessary to describe not just the passage from worker to disable, death or withdrawal, but its necessary to identify passages among the state disable to the death for instance, or even to the state j to *end of the contract*. In insurance in fact it is of interest to see the financial impact of these transitions. Multiple state model has proved to be an appropriate model for an insurance policy in which the payment of benefits or premiums depends on being in a given state or moving between a given pair of states at a given time.
We will discuss so this models to describe the probabilities of moving among these various states, where it is also possible to add movements back and forth between two states.
The most frequently used multiple state transition model is the *Markov process*, it can be formulated in both continuous time framework of discrete time framework; we introduce here the concept of Markov chain to describe probabilities of transitions among states.

3.3.1 Markov Chain

Let $\{X_t, t \geq 0\}$ be a Markov chain with finite state space $S = \{1, 2, \ldots, m\}$,and X_t denoting the state of the system at time t.
It satisfies the Markov property given by:

$$P\left[X_{t+1} = x_{t+1} \mid X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0\right] = P\left[X_{t+1} = x_{t+1} \mid X_t = x_t\right] \tag{48}$$

as long as the conditional probabilities are defined. The Markov property is usually described as "history independence", this means that the probability distribution of the state of the system at time $t+1$ could depend on the state at time t but does not depend on the states at times $t-1, t-2, t-3, \ldots, 0$. Each X_t is a *discrete random variable* with a set S of possible values. The joint probabilities related to the Markov chain can be expressed in terms of the conditional distribution of X_{t+1} given $X_t = x_t$. We denote the conditional probability $P\left[X_{t+1} = j \mid X_t = i\right]$ by $P_t^{(i,j)}$. Thus,

$$P_t^{(i,j)} = P\left[X_{t+1} = j \mid X_t = i\right] \tag{49}$$

is the *one-step transition probability* from state i at time t to state j at time $t+1$.

When the set of one step-transition probabilities depends on t, the Markov chain is defined as *non-homogeneous*, and when it is independent from t, then the Markov chain is known as *homogeneous*. For a finite state space Markov chain, transition probabilities are often described in a matrix notation. Let $P_t = P_t^{(i,j)}$ denote the matrix of transition probabilities from state i at time t to state j at $t+1$, $P_t^{(i,j)}$ being the (i,j) th element of P_t. So, with a state space $S = (1,2,3)$consisting of three elements, Q_t is a 3×3 matrix, called also *transition probability matrix*:

$$\mathbf{P_t} = \begin{pmatrix} P_t^{(1,1)} & P_t^{(1,2)} & P_t^{(1,3)} \\ P_t^{(2,1)} & P_t^{(2,2)} & P_t^{(2,3)} \\ P_t^{(3,1)} & P_t^{(3,2)} & P_t^{(3,3)} \end{pmatrix}$$

Suppose now a generic pension funds with 3 decrements (insured events) and an age-retirement requirement γ, suppose that $j = 1$ denotes the state in which the member of age (x) is still a worker, $j = 2$ denotes the disability pension, $j = 3$ a retirement pension and $j = 4$ the state representing a pension to survivors. Then the transition matrix P_t can be specified in terms of its elements as:

$$\mathbf{P_t} = \begin{pmatrix} P_t^{(1,1)} & P_t^{(1,2)} & P_t^{(1,3)} & P_t^{(1,4)} \\ P_t^{(2,1)} & P_t^{(2,2)} & P_t^{(2,3)} & P_t^{(2,4)} \\ P_t^{(3,1)} & P_t^{(3,2)} & P_t^{(3,3)} & P_t^{(3,4)} \\ P_t^{(4,1)} & P_t^{(4,2)} & P_t^{(4,3)} & P_t^{(4,4)} \end{pmatrix}$$

where:

$$P_t^{(j,j)} = p_{x+t}^{j}$$
$$P_t^{(1,3)} = \begin{cases} 0 & \text{if} \quad x+t < \gamma \\ q_{x+t}^{1,3} > 0 & \text{if} \quad x+t \geq \gamma \end{cases}$$
$$P_t^{(i,j)} = \begin{cases} 0 & \text{if} \quad i = 2, \quad j = 3, \quad i \neq j \\ q_{x+t}^{i,j} > 0 & \text{if} \quad i = 1,2,3 \quad j = 4, \quad i \neq j \end{cases}$$

we remark that, according to international actuarial notation, q denotes the failure probabilities, while p denotes the so-called success probabilities.

3.3.2 Markovian process transition model

Let $\{Y(t), t \geq 0\}$ be a Markov process with finite state space $S = \{1, 2, \ldots, m\}$ $Y(t)$, denoting the state of the system at time t. Let's assume that it satisfies the *Markov property*:

$$P[Y(s+t) = j \mid Y(s) = i, Y(u) = x(u), 0 \leq u \leq s] = P[Y(s+t) = j \mid Y(s) = i]$$

So, the future of the process, after time s, depends only at the state at time x and not on the history of the process up to time s, where each $Y(t)$ is a discrete random variable with set S as the set of possible values. In view of Markov property, the probability structure depends on the transition probabilities is defined as:

$$p_{ij}(s, s+t) = P[X(s+t) = j \mid X(s) = i]$$

$$\text{with } \sum_{j=1}^{k} p_{ij}(s, s+t) = 1 \text{ for all } s, t \geq 0$$

We remark that, If $p_{ij}(s, s+t)$ depends only on t, then it is a time-homogeneous Markov process; otherwise we speak about non-homogeneus Markov process. In a context such the pension funds or the life insurance, the event $Y(t) = j$ mean that an individual of age x is in state j at age $x+t$. In such way we can represent the life of a contract through the Markovian process $\{X_t, t \geq 0\}$. We introduce now the notation for transition probabilities of a Markov model for a pension fund context:

$${}_tq_x^{i,j} = P[Y(x+t) = j \mid Y(x) = i]$$

that represents the probability that an individual currently aged x who is currently in state i, will be in state j at age $x+t$, and

$${}_tp_x^{i} = P[Y(x+t) = i \mid Y(x) = i]$$

namely, the probability that the individual that is currently in state i at age x will be in state i at age $x+t$.

Notice that if we admit backward transitions, we should distinguish between $p_x^{\bar{i}}$ and p_x^{i}:

${}_tp_x^{\bar{i}}$ is the probability that the process/individual does not leave state i between ages x and $x+t$, while ${}_tp_x^{i}$ is the probability that the process/individual is in state i at age $x+t$, being in state i at age x.

Anyway, for any individual state which either can never be left or can never be re-entered once it has been left, these two probabilities are equivalent and so the following inequality is always true:

$${}_tp_x^{\bar{i}} \leq {}_tp_x^{i}.$$

4 Multi-state Model for Pension Funds

In this section we describe the multi-state model set up to perform the analyzes that we will discuss in the last chapter, we will also describe the assumptions used for the construction of the pension fund as well as the theoretical and application results used to calculate the benefits and contributions of the insured parties.

4.1 Model assumption

We assume that the pension fund is characterized by the following states, denoted with $i = 0, 1, \ldots, 5$. We define with $i = 1$ the state that contain active members, namely, workers paying contributions into the pension scheme.
At the second level in our multi-state model, we have all the direct benefits that can be obtained in case of exit for different cause. In particular, we have:

- $i = 2$: Pensioners for Disability retirement;
- $i = 3$: Pensioners for Age Retirement;

At the third level we have the pensions concerning the relatives (eligible for benefits) of a deceased worker or pensioner who did not hold a pension but who, at the date of death, met the stipulated insurance and contribution requirements.

- $i = 4$: Survivors of a deceased worker;
- $i = 5$: Survivors of a deceased pensioner (disability or age retiriment).

Finally, we define an absorbing state $i = 0$, that considers all members that exit from the pension fund without any benefit. According to the markovian notation, an absorbing state is a state that, once entered, cannot be left.

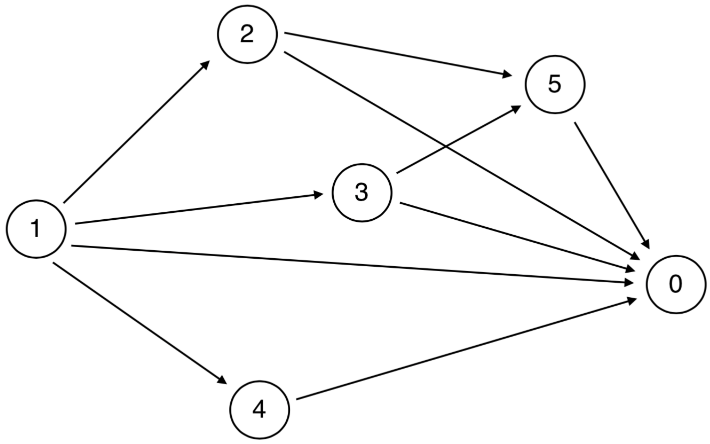

Figure 1: Graphical representation of the states with links.

We also assume that each member can move in the hierarchical structure as represented in the picture above:

- *Workers* (state 1) can only move to state 2 (disability retirement), state 4 (indirect pension to survivors or at state 3 once the age of requirement is reached or state 0;
- *Pensioners* (state 3 and state 2) can only move to state 5 (survivors of pensioners) or state 0;
- *Survivors of workers* (state 4) can move only through the absorbing state 0.

Furthermore for simplicity we assume that backward transitions are not possible, so the disability state is only *permanent*; each active workers move to the state 3 once the requirement to be pensioner is met (age of 68).
Each member should stay at least one year in a state before to exit and the passages between states occurs at the end of the year. In the computation of

benefits relative to survivors only the presence of a widow/widower has been considered in order to simplify the computations.
For what concern instead the notation used and the assumption related to the cohorts considered are:

- the pension scheme regards a cohort of members, that enters both the labour market and the fund at age x (entry-age). We assume that the entry-age is bounded as follows: $\alpha \leq x < \nu$, where α is the youngest entry and ν is the max aged people of the cohort.[7]
- β is the age of pensioning for old age retirement and is set to 68.
- t denotes the years of contribution as an active member (state 1). In other words, it represents the seniority in the state 1. We have $t_3 = \beta - x$.
- τ denotes the seniority in the second order groups (state 2 and 3). We have $\tau < \omega - x - t$ where ω is the maximum age reachable.
- η denotes the seniority in the state 4 and state 5. We have $\eta < \omega - x - t - \tau$.[8]
- c, denotes the difference of age between worker/pensioner and widow/widower, it is assumed to be $+3$ for the survivor of female workers and -3 for survivors of male workers.

4.2 Expected number of People

Our first aim is to estimate the expected number of people that belong to each group at the generic future time m; this will be later useful in order to compute

[7] This setup has been made in order to consider only age-retirement pensioners with a minimum seniority of 20 years; in general the following relation must hold: $\nu < \beta$

[8] We have assumed in inequalities that the age of the survivor at death is equal to the age of the deceased worker/pensioner.

the reserves and to compare some results.

We assume to be at time 0 and to know the composition (age and sex) of entrants in the fund, we suppose also to know the future entrants, which will be an increasing number with the passage of the years. We aim at estimating the expected number of people $L^{(m)} = \sum_{i=0}^{5} L_i^{(m)}$ where $L_i^{(m)}$ represents the expected number of people that belong to the group i at a generic time $m > 0$. In general, pension funds are interested in more detailed information, because both the transition probability and the benefit are related to the characteristic of the individual. Hence, we aim at estimating $L_{1,(x,t)}^{(m)}, L_{i,(x,t,\tau)}^{(m)}($ with $i = 2, 3), L_{j,(x,t,\tau,\eta)}^{(m)}$, with $i = 4, 5$ respectively.

For the *state 1* we have:

$$L_{1,(x,t)}^{(m)} = N_{1,(x,0)}^{(m-t)} \cdot {}_t p_{[x]}^{1} \qquad \text{with } 0 \leq t \leq m \tag{50}$$

where $N_{1,(x,0)}^{(m-t)}$ represents the expected number of entrants at time $m-t$ in the group 1 (i.e. new workers that join the pension fund), and $[x]$ represents the age of entrants in the state. We wants to highlight that $t = m$ for the initial cohort, while will be lower for the new entrants; this holds also for the further results.

About the *state 2* we have:

$$L_{2,(x,t,\tau)}^{(m)} = N_{1,(x,0)}^{(m-t-\tau)} \cdot {}_{t-1} p_{[x]}^{1} \cdot q_{[x]+t-1}^{1,2} \cdot {}_\tau p_{[x+t]}^{2} \qquad \text{with } t + \tau \leq m \tag{51}$$

Where $q_x^{1,2}$ is the probability of transition from state 1 to state 2, namely the probability of a work to become disable at age x. Here the people in state 2 at time m are the people who are entered in the state 1 $(m - t - \tau)$ years before, have passed $t - 1$ as workers and then have passed to disability state, remaining there for τ years.

For the *state 3* we have:

$$L_{3,(x,t,\tau)}^{(m)} = N_{1,(x,0)}^{(m-(\beta-x)-\tau)} \cdot {}_{\beta-x} p_{[x]}^{1} \cdot {}_\tau p_{[\beta]}^{3} \qquad \text{with } (\beta - x) + \tau \leq m \tag{52}$$

Where $t = \beta - x$ and the passage to the state 3 for the worker is conditional to the reach of the age of retirement β.
The *State 4* people are calculated as:

$$L_{4,(x,t,\eta)}^{(m)} = N_{1,(x,0)}^{(m-t-\eta)} \cdot {}_{t-1}p_{[x]}^{1} \cdot q_{[x]+t-1}^{1,4} \cdot \theta_{x+t}^{f} \cdot {}_{\eta}p_{[x+c+t]}^{4} \qquad \text{with } t+\eta \leq m \tag{53}$$

where θ_x^f is the probability to leave a family at age x and c is the difference of age between the worker and the survivor.
Finally the *State 5* number of people is:

$$\begin{aligned} L_{5,(x,t,\tau,\eta)}^{(m)} = N_{1,(x,0)}^{(m-t-\tau-\eta)} &\cdot {}_{t-1}p_{[x]}^{1} \cdot q_{[x]+t+-1}^{1,2} \cdot \\ &{}_{\tau-1}p_{[x+t]}^{2} \cdot q_{[x]+t+\tau-1}^{2,5} \cdot \theta_{x+t+\tau}^{f} \cdot {}_{\eta}p_{[x+c+t+\tau]}^{5} \\ &+ N_{1,(x,0)}^{(m-(\beta-x)-\tau-\eta)} \cdot {}_{\beta-x}p_{[x]}^{1} \cdot {}_{\tau-1}p_{[\beta]}^{3} \cdot \\ &q_{[x]+(\beta-x)+\tau-1}^{3,5} \cdot \theta_{\beta+\tau}^{f} \cdot {}_{\eta}p_{[\beta+c+\tau]}^{5} \end{aligned} \tag{54}$$

Where $q_x^{2,5}$ and $q_x^{3,5} = q_x^{1,4}$ are the probabilities of death of recipients of respectively, old-age pension and disability pension.

4.3 Capitalization coefficients

In order to price the participation to the *Defined Benefit* pension scheme and set aside the right amount of resources to pay the retirement or disability salary, we need the *expected present value* (EPV) of future salaries and benefits for each member enrolled, related to the entire lifetime of them.
To compute the EPV of future cash-flows for each member we will use the so called *capitalization coefficients* which are basically actuarial annuities appropriately modified to take account of inflation rate, development of salary, transition probabilities, as well as survival probability and discount rate. For the sake of simplicity we assume here that: x is the age of the memeber at the

valuation date, transition probabilities are constant over time, and flat rates. We need further to define:

- s denotes the salary growth rate, such that, naming w the yearly salary $w_t = w_{t-1} \cdot (1+s)$, it embodies both inflation rate changes and career development;
- i denotes the first order basis rate of revaluation of pensions over time;
- j denotes the first order basis rate of return used to discount benefits and reserves;
- $\frac{t}{k}$ denotes the *replacement rate*, it is a function of the year of seniority in the state 1 (t) and a defined ex-ante rate of return k.[9];
- we denote with x the age of the member at the valuation date;
- lastly, for the sake of simplicity, we need to remark that in the further computation we will consider only pure benefit, no expenses of liquidation or management are involved the computation of cash-out flows; consequently we will voluntary not considered expenses loading and expense reserves in the further sections.

Thus, we can classify different levels of capitalization coefficients:

1. *First Type annuity*: this coefficient measures the expected present value of future unit payments (salaries or benefits) for a specific member that belongs to a group as long as he remains in the same group;
2. *Second Type annuity*: this coefficient measures the expected present value of future unit payments (benefits), paid after the member moves to another group as long as he remains in the new group;

[9]The replacement rate represents the portion of the first pension with respect of the function of salary computed the year before the retirement, in it simplest case (last salary function) we have: $\frac{t}{k} = \frac{first\ benefit}{last\ salary}$

3. *Third Type annuity*: this coefficient measures the expected present value of future unit payments (benefits), paid after two group transitions and until he remains in the last group.

We introduce in the following sections the different levels of annuities divided by states.

4.3.1 First type annuities

We denotes with $\ddot{a}_x^i$ the first type annuity relative to the state i, such annuity compute the EPV (benefit or salary) of a subject which belong to the group i as long as he remains in the same group.
For the *State1*:

$$\ddot{a}_x^1 = \sum_{t=0}^{\beta-x-1} {}_tp_x^1 \left(\frac{1+s}{1+j}\right)^t = \sum_{t=0}^{\beta-x-1} \frac{{}_tp_x^1}{(1+j)^t} \cdot \frac{w_t}{w_0} \tag{55}$$

The relation holds since, as seen previously in the assumptions: $w_t = w_0 \cdot (1+s)^t$.
For the *State 2* we have:

$$\ddot{a}_x^2 = \gamma \cdot \sum_{h=0}^{\omega-x-1} {}_hp_x^2 \left(\frac{1+i}{1+j}\right)^h. \tag{56}$$

Where γ is the percentage of the accrued pension paid as disability retirement, while p_x^2 is the probability of survivorship for a member receiving a disability pension.
Instead about the *State 3* we have:

$$\begin{aligned} \ddot{a}_\alpha^3 = {}_{(\beta-\alpha)}E_\alpha^1 \cdot \frac{\beta-\alpha}{k} \sum_{h=0}^{\omega-\beta-1} {}_hp_\beta^3 \left(\frac{1+i}{1+j}\right)^h = \\ \frac{{}_{(\beta-\alpha)}p_\alpha^1}{(1+j)^{(\beta-\alpha)}} \cdot \frac{\beta-\alpha}{k} \cdot \sum_{h=0}^{\omega-\beta-1} {}_hp_\beta^3 \left(\frac{1+i}{1+j}\right)^h, \end{aligned} \tag{57}$$

This first type annuity has been modified introducing an actuarial discount factor in order to take into account the time value between entering state 1 (worker) and the retirement year. Indeed both the probabilities p^1_α, ${}_{(\beta-\alpha)}p^1_\alpha$ are a function of the entry age α and retirement age β.

Lastly, for what concern the survivor's states ($f = 4, 5$) we have used a simplified version of the family annuity coefficient that consider only the survival probabilities of the widow/widower, with the age x appropriately modified by the parameter c, and the probability to leave a family discussed above:

$$\ddot{a}^f_x = K \cdot \theta^f_x \sum_{\eta=0}^{\omega-(x+c)-1} {}_\eta p^w_{x+c} \left(\frac{1+i}{1+j}\right)^\eta , \tag{58}$$

Where K is the percentage of the pension that is paid to the survivors. For the benefits provided by the INPS, these percentages are equal to 60% in case the insured leave only a wife/husband, 70% if the survivor is a child, 80% if remain two components and 100% if three or more. While the term p^w_{x+c} represents the the probability of the widow/widower of still being entitled to receive the survivor's pension at age $x+c$. In order to compute the probability p^w_{x+c} are typically used both likelihood to have a new marriage and the survival probability of the spouse, such as: $p^w_{x+c} = (1 - q^{newmarriage}_{x+c} - q_{x+c})$.

4.3.2 Second type annuities

With the second type annuity we consider the expected present value of future benefits, after one transition and until the next transition, which is according to our framework the death of the member. We can distinguish between four second type annuities which are useful in our model: $\ddot{a}^{(1,2)}_x$, $\ddot{a}^{(1,4)}_x$, $\ddot{a}^{(2,5)}_x$, $\ddot{a}^{(3,5)}_x$. Though not all of them will be used *directly* to compute benefits, but will be back on this later.

The first annuity of type two is necessary to compute the expected present value of a worker aged x at valuation date, which later will be entitled for a disability pension, until the end of the contract:

$$\begin{aligned}\ddot{a}_x^{(1,2)} = \sum_{t=0}^{\beta-x-1} {}_hp_x^1 \cdot q_{x+t}^{1,2} \cdot \left(\frac{1+s}{1+j}\right)^{(t+1)} \cdot \frac{t+1}{k} \cdot \\ \gamma \cdot \sum_{\tau=0}^{\omega-(x+t+1)-1} {}_\tau p_{x+t+1}^2 \cdot \left(\frac{1+i}{1+j}\right)^{(\tau)} .\end{aligned} \tag{59}$$

This last can be rewritten by recalling our definition of $\ddot{a}_x^2$ at eq. 56:

$$\ddot{a}_x^{(1,2)} = \sum_{t=0}^{\beta-x-1} {}_hp_x^1 \cdot q_{x+t}^{1,2} \cdot \left(\frac{1+s}{1+j}\right)^{(t+1)} \cdot \frac{t+1}{k} \cdot \ddot{a}_{x+t+1}^2 . \tag{60}$$

The next annuity concerns a worker which dies during the contribution period and leave (probably) a family which is entitled to receive an indirect pension, until the death of survivors or the loss of rights:

$$\begin{aligned}\ddot{a}_x^{(1,4)} &= \sum_{t=0}^{\beta-x-1} {}_tp_x^1 \cdot q_{x+t}^{1,4} \cdot \left(\frac{1+s}{1+j}\right)^{(t+1)} \cdot \frac{t+1}{k} \cdot K \\ &\cdot \theta_{x+t+1}^f \cdot \sum_{\eta=0}^{\omega-(x+c+t+1)-1} {}_\eta p_{x+c+t+1}^w \cdot \left(\frac{1+s}{1+j}\right)^{(\eta)} \\ &= \sum_{t=0}^{\beta-x-1} {}_tp_x^1 \cdot q_{x+t}^{1,4} \cdot \left(\frac{1+s}{1+j}\right)^{(t+1)} \cdot \frac{t+1}{k} \cdot \ddot{a}_{x+t+1}^f .\end{aligned} \tag{61}$$

The following annuity instead concern the expected present value relative to pension recipients who die during and leave the indirect pension to survivors: The second type annuity concerning EPV of a disability pensioner is:

$$\begin{aligned}\ddot{a}_x^{(2,5)} &= \sum_{\tau=0}^{\omega-x-1} {}_\tau p_x^2 \cdot q_{x+\tau}^{2,5} \cdot \left(\frac{1+i}{1+j}\right)^{(\tau+1)} \cdot K \cdot \theta_{x+\tau+1}^f \\ &\cdot \sum_{\eta=0}^{\omega-(x+c+\tau+1)-1} {}_\eta p_{x+c+\tau+1}^w \cdot \left(\frac{1+i}{1+j}\right)^{(\eta)} \\ &= \sum_{\tau=0}^{\omega-x-1} {}_\tau p_x^2 \cdot q_{x+\tau}^{2,5} \cdot \left(\frac{1+i}{1+j}\right)^{(\tau+1)} \cdot \ddot{a}_{x+\tau+1}^f ,\end{aligned} \tag{62}$$

Instead the annuity of second type concerning the EPV relative to a retirement pensioner is:

$$
\begin{aligned}
\ddot{a}_{\alpha}^{(3,5)} = & \sum_{\tau=0}^{\omega-x-1} {}_{\tau}p_{\beta}^{3} \cdot q_{\beta+\tau}^{3,5} \cdot \left(\frac{1+i}{1+j}\right)^{(\tau+1)} \cdot K \cdot \theta_{\beta+\tau+1}^{f} \\
& {}_{(\beta-\alpha)}E_{\alpha}^{1} \cdot \frac{\beta-\alpha}{k} \cdot \sum_{\eta=0}^{\omega-(\beta+c+\tau+1)-1} {}_{\eta}p_{\beta+c+\tau+1}^{w} \cdot \left(\frac{1+i}{1+j}\right)^{(\eta)} \\
& = {}_{(\beta-\alpha)}E_{\alpha}^{1} \cdot \frac{\beta-\alpha}{k} \sum_{\tau=0}^{\omega-\beta-1} {}_{\tau}p_{\beta}^{3} \cdot q_{\beta+\tau}^{3,5} \cdot \left(\frac{1+i}{1+j}\right)^{(\tau+1)} \cdot \ddot{a}_{\beta+\tau+1}^{f}
\end{aligned} \tag{63}
$$

In the next section we see lastly the third type annuities, which will compound by using the results obtained until now.

4.3.3 Third type annuities

Lastly we have the third type annuities. Those capitalization coefficients calculate the expected present value in case of three transitions. According to our model the next annuity, that will be necessary in order to calculate the EPV of the indirect pension to survivors of a disability retired member, is:

$$
\begin{aligned}
\ddot{a}_{x}^{(1,2,5)} = & \sum_{t=0}^{\beta-x-1} {}_{h}p_{x}^{1} \cdot q_{x+t}^{1,2} \left(\frac{1+s}{1+j}\right)^{(t+1)} \cdot \frac{t+1}{k} \cdot \\
& \gamma \cdot \sum_{\tau=0}^{\omega-(x+t+1)-1} {}_{\tau}p_{x+t+1}^{2} \cdot q_{x+t+\tau+1}^{2,5} \left(\frac{1+i}{1+j}\right)^{(\tau+1)} \cdot \\
& K \cdot \theta_{x}^{f} \sum_{\eta=0}^{\omega-(x+t+\tau+1+c)} {}_{\eta}p_{x+c+t+\tau+2}^{w} \left(\frac{1+i}{1+j}\right)^{\eta},
\end{aligned} \tag{64}
$$

which can be rewritten as:

$$\ddot{a}_x^{(1,2,5)} = \sum_{t=0}^{\beta-x-1} {}_h p_x^1 \cdot q_{x+t}^{1,2} \left(\frac{1+s}{1+j}\right)^{(t+1)} \cdot \gamma \cdot$$

$$\sum_{\tau=0}^{\omega-(x+t+1)-1} {}_\tau p_{x+t+1}^2 \cdot q_{x+t+\tau+1}^{2,5} \left(\frac{1+i}{1+j}\right)^{(\tau+1)} \cdot \ddot{a}_{x+t+\tau+2}^{(f)} \tag{65}$$

$$= \sum_{t=0}^{\beta-x-1} {}_h p_x^1 \cdot q_{x+t}^{1,2} \left(\frac{1+s}{1+j}\right)^{(t+1)} \cdot \frac{t+1}{k} \cdot \ddot{a}_{x+t+1}^{(2,5)}$$

Finally we want to highlight that in the next chapter we have studied different way to compute the first pension, still according with the defined benefit scheme. In particular in the annuities seen until here, the first benefit paid (b_β), is calculated according to a defined benefit scheme with a salary function equal to the last salary $w_{\beta-1}$, such as:

$$b_\beta = \left(\frac{t+1}{k}\right) \cdot w_{\beta-1} \tag{66}$$

Though, as mentioned above, other function of the salary can be used, such as the average of last n salaries or general average of salaries during the whole career. For completeness we report here the third type annuity, with the function $\bar{w}$ used in latter moment, (together with the other annuities), to make some sensibility analysis:

$$\ddot{a}_x^{(1,2,5)} = \sum_{t=0}^{\beta-x-1} \frac{{}_h p_x^1 \cdot q_{x+t}^{1,2}}{(1+j)^{(t+1)}} \cdot \frac{\bar{w}}{w_0} \cdot \frac{t+1}{k} \cdot \ddot{a}_{x+t+1}^{(2,5)} \tag{67}$$

Where as defined previously, $\frac{\bar{w}}{w_0}$ denotes the average salary over the all career of the member on the initial salary w_0.

In the paragraph 4.6 we will use the expected present values described in this section in order to compute the EPV of both benefits and salaries for the entire cohort of workers entering in the fund in the same future calendar year m; and consequently to compute, using those expected present value, the *premium rate.*

4.4 Transformation coefficients

In a capitalization structure with *Defined Contributions* the amount of premium paid to the pension fund is defined ex-ante. According to this framework an accumulation plan is set up, the resources paid by the member are pooled in order to be invested in the market by the fund manager. During the contribution period the capital accumulated is revaluated according to the return recognized by the fund, and at the retirement date, if no guarantees are involved, the future benefits are evaluated through the so called *transformation coefficients* (TC). This coefficient basically calculate the portion of capital accrued which will then paid each year as pension benefit. They are derived by the expected value of the future benefits annuities discounted to the year of retirement. Similarly to the annuities seen in the previous chapter, some assumptions are necessary in order to perform the computation, though those assumptions are relative to the payment period only and not to the accumulation phase. Such hypothesis concern: rate of return, rate revaluation of pensions and naturally assumptions relative to the length of life of the insured members.

We describe below in details the transformation coefficients relative to the different cause of decrement.

For what concern the TC relative to the *old age retirement* we have:

$$TC_{\beta}^{3} = \frac{1}{{}^{c}\ddot{a}_{\beta}^{3} + {}^{c}\ddot{a}_{\beta}^{3,5}} \tag{68}$$

where ${}^{c}\ddot{a}_{\beta}^{3}$ and ${}^{c}\ddot{a}_{\beta}^{3,5}$ are respectively a type one annuity and a type two annuity, defined as:

$${}^{c}\ddot{a}_{\beta}^{3} = \sum_{\tau=0}^{\omega-\beta-1} {}_{\tau}p_{\beta}^{3} \left(\frac{1+i}{1+j}\right)^{\tau}, \tag{69}$$

$$ {}^{c}\ddot{a}_{\beta}^{3,5} = \sum_{\tau=0}^{\omega-\beta-1} {}_{\tau}p_{\beta}^{3} \cdot q_{\beta+\tau}^{3,5} \cdot \left(\frac{1+i}{1+j}\right)^{(\tau+1)} \cdot \ddot{a}_{\beta+\tau+1}^{f}.[10] \tag{70} $$

The *disability retirement* TC instead is defined as:

$$ TC_x^2 = \frac{1}{{}^{c}\ddot{a}_x^2 + {}^{c}\ddot{a}_x^{2,5}} \tag{71} $$

where the annuities are defined in the following way:

$$ {}^{c}\ddot{a}_x^2 = \sum_{h=0}^{\omega-x-1} {}_{\tau}p_x^2 \left(\frac{1+i}{1+j}\right)^{\tau}, \tag{72} $$

$$ {}^{c}\ddot{a}_x^{2,5} = \sum_{\tau=0}^{\omega-x-1} {}_{\tau}p_x^2 \cdot q_{x+\tau}^{2,5} \cdot \left(\frac{1+i}{1+j}\right)^{(\tau+1)} \cdot \ddot{a}_{x+\tau+1}^{f}. \tag{73} $$

Finally, we have the TC relative to *indirect pension* of deceased workers:

$$ TC_x^4 = \frac{1}{{}^{c}\ddot{a}_x^4}, \tag{74} $$

where ${}^{c}\ddot{a}_x^4$ is the type one annuity relative to survivors, which, according to the simplified version described in eq. 58 considers only the likelihood to keep the pension rights for a widow/widower:

$$ {}^{c}\ddot{a}_x^4 = K \cdot \sum_{\eta=0}^{\omega-(x+c)-1} {}_{\eta}p_{x+c}^{w} \left(\frac{1+i}{1+j}\right)^{\eta}. \tag{75} $$

4.5 Expected present values for a portfolio of contracts

Once the framework relative to the expected present value of benefits and salaries has been defined[11], we can combine them with some assumptions relative to the number of new entrants at each future calendar year $m = 0, 1, 2, \ldots$ and the initial salary w_0^m, in order to compute the total expected present value

[10] the family capitalization coefficient is the same seen at eq. 58

[11] See section 4.3

of the whole cohort of entrants. Once the EPVs relative to each year cohort of entrants are computed, those results can be further combined among different cohorts in order to have a prospect of the future development of the pension fund, at least in expectation terms.

For the moment we consider a specific cohort of workers, entering in the fund at the end of time m, with different ages x and a equal initial salary w_0^m.

We denote with $S^m_{(x,0)}$ the expected present value of salaries of a cohort of insured workers, of age x and seniority 0 at time m. We can write this expected present value as:

$$S^m_{(x,0)} = N^m_{(x,0)} \cdot w_0^m \cdot \ddot{a}_x^1. \tag{76}$$

The factor $N^m_{(x,0)}$ represents the number of new entrants in the fund at time m, with age x and seniority 0, while the annuity factor $\ddot{a}_x^1$ is the one discussed in the eq. 55.

The expected present value of benefits, relative to the whole cohort of new entrants can be instead, computed as the sum of the following three components:

$$O^m_{2,(x,0)} = N^m_{(x,0)} \cdot w_0^m \cdot \left[\ddot{a}_x^{(1,2)} + \ddot{a}_x^{(1,2,5)}\right], \tag{77}$$

$$O^m_{3,(x,0)} = N^m_{(x,0)} \cdot w_0^m \cdot \left[\ddot{a}_x^{3} + \ddot{a}_x^{(3,5)}\right], \tag{78}$$

$$O^m_{4,(x,0)} = N^m_{(x,0)} \cdot w_0^m \cdot \ddot{a}_x^{(1,4)}. \tag{79}$$

The eq. 77 concerns the EPV of disability benefits and indirect pension paid to survivors of disability pension recipients; here we can notice that there are a type two annuity and a third type annuity since the condition to receive a disability pension is to make exactly two transitions, while, in order to be in the state 5, three transitions are necessary. The same does not hold though in eq. 78, and this is because of the way we set up the annuities relative to the state 3, namely in fact we implied the transition between state 1 and state 3 directly in the type one annuity. The equation 79 is instead relative to survivors of worker members, since the decrement concern the death of the member in state 1, only one passage is involved and so a type one annuity is present here.

If one would be interested in knowing the total expected present value of benefits relative only to the state 5 (survivors of pensioners), we have:

$$O^{m}_{5,(x,0)} = N^{m}_{(x,0)} \cdot w^{m}_{0} \cdot \left[\ddot{a}^{(3,5)}_{x} + \ddot{a}^{(1,2,5)}_{x} \right] . \tag{80}$$

Now that the expected present values of future cash flows have been determined, we can move a step forward in order to compute the pure *premium rates* for the cohorts of new entrants.

4.6 Premiums

We consider again a specific cohort of workers, entering in the fund at time m, with different ages x.

According to the fully funded methodology, we can distinguish between three different levels of solidarity in the computation of premium rates[12], those methods vary according to the type and specific regulation of the pension fund; we can have: Individual contribution rate, Cohort contribution rate and Multi-cohort contribution rate.

[12] We remark that in the following computation we will consider only the pure part of premium rates, no expense loading so will be considered for now.

An individual contribution rate can be defined at the entry age x as:

$$P_{(x,0)}^{(m)} = \frac{\sum_{i=2}^{4} O_{i,(x,0)}^{(m)}}{S_{(x,0)}^{(m)}} = \frac{\ddot{a}_x^{3(m)} + \ddot{a}_x^{(3,5)(m)} + \ddot{a}_x^{(1,4)(m)} + \ddot{a}_x^{(1,2)(m)} + a_x^{(1,2,5)(m)}}{\ddot{a}_x^{(1)(m)}} \tag{81}$$

The cohort contribution rate instead is:

$$P^{(m)} = \frac{\sum_{x=\alpha}^{\nu} \sum_{i=2}^{4} O_{i,(x,0)}^{(m)}}{\sum_{x=\alpha}^{\nu} S_{(x,0)}^{(m)}} = \frac{\sum_{x=\alpha}^{\nu} P_{(x,0)}^{(m)} S_{(x,0)}^{(m)}}{\sum_{x=\alpha}^{\nu} S_{(x,0)}^{(m)}} \tag{82}$$

Where as denoted at the inception of the chapter, α and ν are respectively the minimum age and the maximum age of the cohort.

According to this formulation of the premium, with respect with the individual approach, we have an higher degree of solidarity, in fact here we consider a same cohort compound by different age members. What follows is that higher individual premium rates will be naturally compensated by lower premium rates.

Finally an ulterior level of solidarity can be reached by aggregating different cohorts and compute an unique contribution rate:

$$P^{[0,k]} = \frac{\sum_{m=0}^{k} \left(\sum_{x=\alpha_m}^{\nu_m} \sum_{i=2}^{5} O_{i,(x,0)}^{(m)} \right) v^m}{\sum_{m=0}^{k} \left(\sum_{x=\alpha_m}^{\nu_m} S_{(x,0)}^{(m)} \right) v^m} = \frac{\sum_{m=0}^{k} P^{(m)} \sum_{x=\alpha_m}^{\nu_m} S_{(x,0)}^{(m)} v^m}{\sum_{m=0}^{k} \left(\sum_{x=\alpha_m}^{\nu_m} S_{(x,0)}^{(m)} \right) v^m}, \tag{83}$$

where α_m and ν_m are respectively the minimum age and the maximum age of the cohort m.

In the extreme case where $k \to \infty$, we have the *general average premium*. It represents the constant contribution rate applicable infinitely, theoretically guaranteeing that scheme expenditure can always be covered by the contributions collected and the funds accumulated in the reserve. It is calculated by dividing the present value of all future benefits (minus the reserve existing at the valuation date by the present value of all future contributory earnings.

4.7 Mathematical Reserve

In order to compute the mathematical reserve we need before to define the formulation relative to the total expected value of benefits and contributions relative to each state at each future time $m = 0, 1, \ldots, \omega - 1$, where here with $\omega - 1$ we denote the last calendar year where benefits related to existing members at time 0 are paid.

4.7.1 Expected value of cash flows

The expected value of future cash flows can be divided in inflows and outflows, the inflows naturally consists in the contribution paid by the members, so technically the fund will have a positive amount of inflows (relative to the existing working members at time 0), until the last worker will retire or die. Typically this moment, according to our model framework coincides with the year in which the younger members of the cohort reach the retirement age β. For what concern the outflows instead, those have typically an higher duration since are paid until the death of the pensioner or eventually of the survivors, though, since we are considering also disability retirements and survivors of workers, they overlap also at early years with the payment of contributions.

We denote now the functions relative to the total expected value (EV) of salaries, contributions and benefits at future calendar year m. For what concern the EV of salaries for workers of age x and seniority t we have:

$$\bar{S}^{(m)}_{(x,t)} = L^{(m)}_{1,(x,t)} \cdot w^m_t \tag{84}$$

where $L^{(m)}_i$ denotes, as we have seen in section 4.2, the number of people belonging to the state i, and w^m_t denote the salary of a worker with a seniority t in the group 1.

According to the previous formula, we can derive the expected total amount of salaries at each year m as:

$$\bar{S}^{(m)} = \sum_{x=\alpha}^{\beta-1} \sum_{t=1}^{\beta-\alpha} \bar{S}^{(m)}_{(x,t)} = \sum_{x=\alpha}^{\beta-1} \sum_{t=1}^{\beta-\alpha} L^{(m)}_{1,(x,t)} \cdot w^m_t \tag{85}$$

and so the expected value of contributions $\bar{C}^{(m)}$ at each future year m are:

$$\bar{C}^{(m)} = \bar{S}^{(m)} \cdot P^{(m)}, \tag{86}$$

obviously similar results can be obtained at individual cohort level:

$$\bar{C}^{(m)}_{(x,t)} = \bar{S}^{(m)}_{(x,t)} \cdot P^{(m)}_{(x,0)}. \tag{87}$$

For what concern instead the different benefit states, we have:

$$\bar{O}^{(m)}_{2,(x,t,\tau)} = L^m_{2(x,t,\tau)} \cdot w^m_t \cdot \frac{t}{k} \cdot \gamma \tag{88}$$

$$\bar{O}^{(m)}_{3,(x,t,\tau)} = L^m_{3(x,t,\tau)} \cdot w^m_t \cdot \frac{t}{k} \tag{89}$$

$$\bar{O}^{(m)}_{4,(x,t,\eta)} = L^m_{4(x,t,\eta)} \cdot w^m_t \cdot \frac{t}{k} \cdot K \tag{90}$$

$$\bar{O}^{(m)}_{5,(x,t,\tau,\eta)} = L^m_{5(x,t,\tau,\eta)} \cdot w^m_t \cdot \frac{t}{k} \cdot K \tag{91}$$

Where (e.g) $L_{2(x,t,\tau)}$ denotes the number of contracts in the state 2 where the age of the disable member is x[13], the seniority reached in state 1 is t, and is in the state 2 since τ years. Since under our assumptions the salary received by workers increases with the increases of the seniority according the career

[13] In $L_{5(x,t,\tau,\eta)}$ the age x is the age the original member of the fund would have if he was still alive.

development and inflation, w_t^m denotes the salary at time m for a subject with a seniority in state 1 of t years. $\frac{t}{k}$ denotes the replacement rate at the year of pensioning, indeed consider the t year of seniority in state 1. Finally, the parameters K and γ are the percentage of accruaed pension paid for disability and survivors.

The total expected value of benefits that will be paid the year m is so:

$$\bar{O}^{(m)} = \sum_{x=\alpha+1}^{\omega-1} \sum_{t=1}^{\beta-\alpha} \sum_{\tau=1}^{\omega-1} \sum_{i=2,3} \bar{O}^{(m)}_{i,(x,t,\tau)} + \sum_{x=\alpha+1}^{\omega-1} \sum_{t=1}^{\beta-\alpha} \sum_{\eta=1}^{\omega-1} \bar{O}^{(m)}_{4,(x,t,\eta)} + \sum_{x=\alpha+1}^{\omega-1} \sum_{t=1}^{\beta-\alpha} \sum_{\tau=1}^{\omega-1} \sum_{\eta=1}^{\omega-1} \bar{O}^{(m)}_{5,(x,t,\tau,\eta)}. \tag{92}$$

4.7.2 Reserves

Assuming to be at time 0, we want now to estimate the expected value of future reserves for each calendar year m. According to the actuarial definition of prospective mathematical reserve we have:

$$V_t = Benefits(t,n) - Premiums(t,n)$$

and so for what concern active members (a) at time 0, we have that the EV of future claims reserve is:

$$V_a^{(m)} = \sum_{r=m}^{\omega-1} \frac{\left(\bar{O}_a^{(r)} - \bar{C}_a^{(r)}\right)}{(1+j)^{r-m}} \tag{93}$$

where $\omega - 1$ still represents the last calendar year where benefits, related to existing members at time 0 are paid and $(1+j)^{r-m}$ is the discounting factor.[14]

The previous formula can be further decomposed in two parts:

[14] As it is clear from the formulation, here we are assuming a constant discount rate j.

$$V_a^{(m)} = V_{aa}^{(m)} + V_{ap}^{(m)} = \sum_{r=m}^{\beta-1} \frac{\left(O_a^{(r)} - \bar{C}_a^{(r)}\right)}{(1+j)^{r-m}} + \sum_{r=m}^{\omega-1} \frac{\left(O_a^{(r)} - \bar{\Theta}_a^{(r)}\right)}{(1+j)^{r-m}} \tag{94}$$

where V_{aa} denotes the component of reserve for people which are still active at time m, while V_{ap} denotes the portion of reserve for people which are active at time 0 but will not be anymore active at the future time m. In particular the first component is compounded by the discounted difference between the expected present value of future benefits for new pensioners in r and the expect value of contributions in r; while the second component by discounted difference between the expected value of benefits paid in r and the expect value of future benefits for new pensioners in r, denoted by $\bar{\Theta}_a$.
Finally, for new entrants in a generic year k, (with $k \leq m$), we have that the expected value of future claims reserve is:

$$V_a^{(m)} = \sum_{r=m}^{\omega-1} \frac{\left(\bar{O}_{a,(k)}^{(r)} - \bar{C}_{a,(k)}^{(r)}\right)}{(1+j)^{r-m}} \tag{95}$$

where here, $\bar{O}_{a,(k)}^{(r)}$ and $\bar{C}_{a,(k)}^{(r)}$ are the expected value of benefits and contributions in the year r related to people that join the fund at time k, $\omega-1$ represents here instead the last calendar year where benefits, related to members that join the fund at time k, are paid.

5 Solvency in Pension Funds

Hand in hand with the insurance business, in the light of recent events, the world of pension funds also needs to be ready to face particularly negative scenarios. The instrument for guaranteeing the solvency of pension funds and consequently also the ultimate objective of protecting the members is (among others) an adequate control of the resources and risks connected with the management of resources and an appropriate regulatory framework that guarantees correct and safe management of the aforementioned risks, of any kind they may be. Another very important tool that is necessary in order to protect policyholders, also regulated by the various legislators, is the transparency and importance of disclosure to stakeholders. To this end, both European and national regulations have been introduced in the last twenty years to make the management of pension funds safer, more transparent, but also harmonized in the European context in order to maintain the stability and competitiveness necessary for the correct development of the business.

5.1 European Regulation Framework

The following section gives a general overview of the latest European reference regulations in terms of guarantee and solvency relating to the business of pension funds. It should be noted though that the legislation relating to these topics is still evolving. Refer therefore to subsequent chapters for comparisons and possible results.

5.1.1 IORP I

Rules concerning the basis for assessing the solvency state of pension funds and the introduction of a first capital requirement have been sets for the first time in Italy in 2013 with the receipt of the European Directive 2003/41/EC, called *IORP I* (Institutions for Occupational Retirement Provision), with the particular objective of guaranteeing a high level of protection for future pensioners, as well as the harmonization of the rules concerning European pension funds, both in terms of disclosure and solvency. In particular, it ensures that pension institutions, which undertake the obligations to cover biometric risks or guaranteed investment or a certain level of guarantee of benefits, hold in order to ensure solvency, additional assets with respect to the technical provisions.

The specific reference rule for the minimum calculation of the additional assets to be held is described in Directive 2002/83/EC, which also regulated the minimum solvency capital requirement for the life insurance business according to the Solvency I regulation. According to the directive, the required margin for a pension fund assuming investment risks is defined as a rate of 4% of the technical reserves plus a rate of 0.3% of the sum at risk if positive; the sum at risk of life products is positive only in the case the event of death is insured, so this last component is considered only if the considered pension fund covers also mortality benefits. In the event that the pension fund does not directly assume the investment risk, COVIP can determine a different percentage compared to 4% and can define technical rules for the determination and calculation of the additional pension fund. In particular, the rules designed by the *IORP I* were not intended to be completely binding on the method of calculating the solvency margin; on the contrary, it is specified that regulators of each member state have the right to established in their territory to hold additional own funds or to establish more detailed rules, provided they are justified from a prudential point of view. Whenever the assets are not sufficient to cover the technical provisions, the fund is required to immediately draw up concrete and realistic recovery plan. Concerning the implementation of this plan, pension funds may be allowed to hold, for a limited period, assets that are insufficient to cover future liabilities. This plan is subject to approval by the COVIP and must indicate the time needed to reintegrate the necessary assets and cover the technical provisions.

It should be noted that Directive 2009/138/EC, better known as Solvency II and entered into force on 1.1.2016, provided some changes made to Article 17 of the *IORP I* Directive. Though we will not investigate further since the changes introduced subsequently do not affect the described methods of calculating the solvency requirement of a pension fund.

5.1.2 IORP II

In March 2011, the European Commission asked EIOPA (European Insurance and Occupational Pension Authority)[15] for an opinion on the revision of the IORP I Directive, to create a new European legislative framework based on risk. This request for further harmonization is mainly due to four causes: first of all, there was a need to facilitate cross-border activities, in fact, despite the occupational pension funds of the European Union benefiting from the principles of free movement and mutual recognition, the social security regulatory obstacles made too expensive to join a pension fund based in a member state other than that in which the work was carried out. Secondly, the financial crisis highlighted the inadequacy of the minimum solvency requirements of the time and the importance of introducing a system for calculating the capital requirement based on the risks the fund is exposed; the third driver concern the development of pension fund business, in particular since the publication of

[15] EIOPA is an institution with statutory powers and legal personality, it is a body of the European Union which has task of supervising the European insurance market. All the insurance supervisory authorities of the European Union participate in it. The key responsibilities of the Authority are supporting the stability of the financial system, the transparency of financial markets and products as well as the protection of policyholders, members and beneficiaries of pension systems.

IORP I, the memberships contribution-based private pension funds increased sharply in the EU area (the same happened in Italy), as described in the previous chapters, in those type of funds the *market risk* and partially those of *longevity* and *inflation* fall on the members, and so was even more necessary to provide a framework able to protect the insured; lastly it was found that the information given to members was often scarce, inadequate and difficult to interpret; this did not allow future members to make a correct and informed investment decision. Furthermore, the disclosure relative to pension schemes was at the time backward when compared to the developments that had already taken place regarding the disclosure on, for example, financial instruments with the introduction of Mifid in 2004.
After two public consultations, EIOPA delivered its final opinion to the European Commission on 15 February 2012, in which it proposed to divide the new *IORP II* Directive into three pillars, similarly to Solvency II:

- *First Pillar*: quantitative requirements for the valuation of assets, liabilities and the Solvency Capital Requirement;
- *Second pillar*: qualitative solvency and governance requirements; rules on the supervisory review process;
- *Third pillar*: transparency requirements regarding the disclosure of information to the supervisory authority, members, and beneficiaries.

Regarding the first pillar, one of the key elements is the revision of the harmonization between the solvency and valuation rules of European pension funds through a new supervision tool: the *Holistic Balance Sheet*. This approach should have allowed pension funds to market evaluate their securities by the presence of adjustment mechanisms to be able to incorporate them into a harmonized balance sheet. The Holistic Balance Sheet also intended to incorporate a capital requirement, such as the *SCR* (Solvency Capital Requirement) and the *MCR* (Minimum Capital Requirement), cores of the *Solvency II* regulation. About pillars II and III, the European Commission intended also to apply similar Solvency II requirements to pension funds on aspects relating to governance, risk management, and disclosure to members and supervisory authorities.

In 2012, EIOPA conducted a first quantitative impact study (QIS) aimed at evaluating the effects of the introduction of the new tools. However, the new provisions were strongly criticized by the recipients and the amount of data collected with the study was considered insufficient since the recipients had not a uniform and harmonized interpretation of the QIS and so the results were not comparable. Following these results, it was announced that the forthcoming IORP II Directive would only cover qualitative and disclosure requirements and that further investigations would be needed to finalize the concept of a holistic balance sheet and risk-based solvency requirement.

5.1.3 Holistic Balance Sheet and Standard Formula

The key element of the quantitative pillar proposed by EIOPA in 2012 is the *Holistic Balance Sheet*. Similarly to what was done with the Solvency 2 directive, the holistic balance sheet was intended to create a harmonized balance sheet common to all pension institutions in the European Union. This tool had been proposed, and evaluated in the QIS, in order to define an instrument of extreme transparency as regards the pension funds of the various member states, which was at the same time comparable and market value evaluated; but it also was a prudential instrument, intending to calculate what would have been the capital requirement to be held.

We remark that at the time of the proposal of *IORP II*, the legislation concerning pension funds was fragmented and not harmonized at all. Therefore, in order to be used as a common tool at the European level, the holistic balance has been structured in such a way as to be able to incorporate and adapt national specificities. In particular was permitted to recognize for instance among the assets, in addition to the value of the investments, the value of the support of the sponsor and the pension protection systems (PPS).

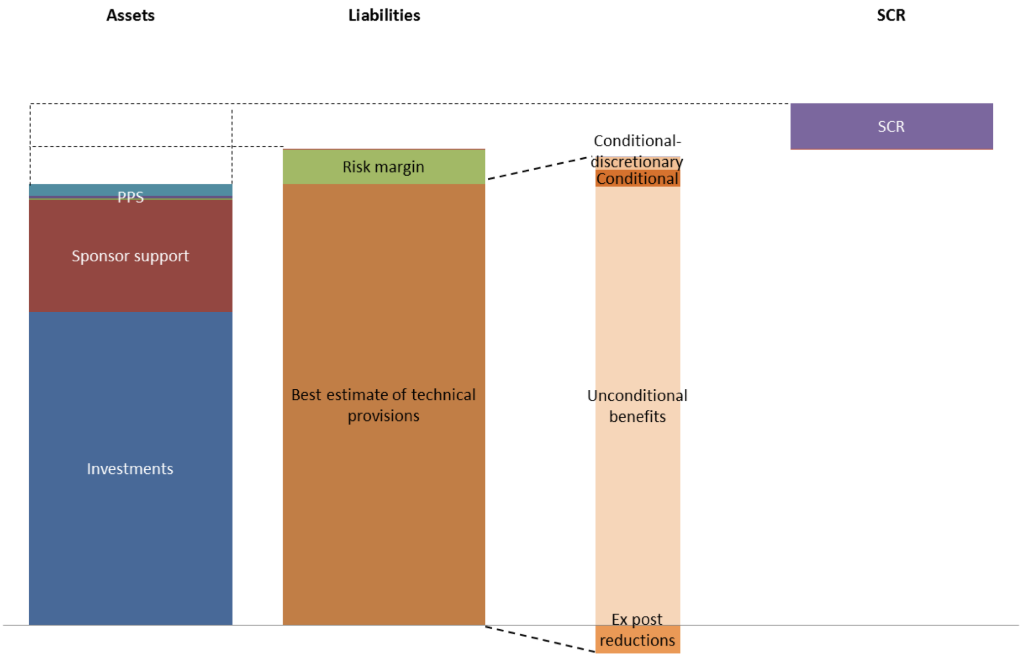

Figure 2: Holistic balance sheet example provided by EIOPA.

For what concerns the liabilities instead, it was further defined that within the valuation of the *best estimate* component three different components of pension benefits were considered and that it was possible under certain conditions to also consider possible reductions in benefits. Regarding the differences between the types of obligations imposed on a fund we have: *Conditional benefits* those benefits granted based on certain objective conditions, and therefore the benefits granted without the fund being able to have any discretion in this regard, such as a benefit whose amount is necessarily linked to variables objectively observed and pre-established in the contract. *Discretionary benefits* those benefits granted under certain subjective conditions; the granting of this lasts, on the other hand, is typically based on financial or demographic developments, without, being a priori contractual connection with them. They are generally granted through periodic decisions taken by the fund on the basis of criteria that are not necessarily formalized. Further in the technical specifications of the following quantitative impact study was further introduced the possibility of a sort of a third *hybrid* category, the *mixed benefits* were defined as those benefits linked to objective events where inside the decisional process have some subjective evaluations.

In a Holistic balance sheet, regardless of the accounting nature of the items, all components must be valued according to a market value approach (full market value balance sheet). This valuation method aims to offer an objective and realistic measure of the financial situation of a pension fund. In this regard, the generally accepted principle is that of 'fair value', which is defined according to the indications of the *International Accounting Standards Board* (IASB) as the price received to sell an asset or paid to transfer a liability in an orderly transaction between market participants on a certain date.

So wherever is possible to determine directly, in a sufficient liquid market, the current market price of an asset or liabilities, can be used directly to determine the *fair value* of the balance sheet item by replicating the portfolio; in this case we speak about *mark-to-market* technique. Though, there is not always a sufficiently liquid market or even a reference market for certain instruments. In these cases, it is necessary to use an appropriate valuation model (*mark-to-model*) capable of quantifying the fair value of the instrument to be valued. The typical case of liabilities that require a valuation model of the type mark-to-model is the *technical reserves* since a reference liquid market does not exist. According to the Holistic balance sheet, the fair value of the mathematical reserves of a pension fund is given by the sum of two components:

the *best estimate*, which is the expected present value of the future inflows net of the expected present value of future cash outflows, the *risk margin*, it is a component which, added to the best estimate, constitutes the amount that the pension fund would ask for as a price to bear the liability in question and the related charges.

Finally, we can arrive at the second main use of the holistic balance sheet. As we have seen above, it is not just a market value for transparency reason, but also a necessary instrument in order to make the computation of what would have been a new risk-based approach for the capital requirement. According to the provisions of EIOPA the capital requirements were intended to be computed according to a *standard formula*. Theoretically, the requirement corresponds to the Value at Risk of the basic own funds of the fund, defined as the surplus of assets over liabilities plus subordinated liabilities, calculated at a confidence level of 99.5% and over a period of one year.[16] In practice, this definition is not directly applied but constitutes the reference point for the calibration of the formula parameters. The capital requirement, called also SCR or Solvency Capital Requirement, according to the standard formula denoted by EIOPA in the technical specification of the Quantitative Assessment of 2016, is calculated through a modular procedure illustrated in figure 3.

Starting from the top, we have the SCR, which is computed by the simple sum between the *Basic SCR* (BSCR) and the other two components, the Operational risk module (defined with Op) and the Adjustments (Adj). The Operational risk module aims to evaluate the potential risk arising from operational failure internal to the funds, while the adjustments concern adjustment relative to the stressed scenarios used to calculate the other submodules; in particular concern the loss-absorbing capacity deriving from unrealized gains in a particularly negative scenario which have to be deducted to the final SCR

[16]refer to section 5.2 for the definition of Value at Risk (Var).

in the name of consistency.

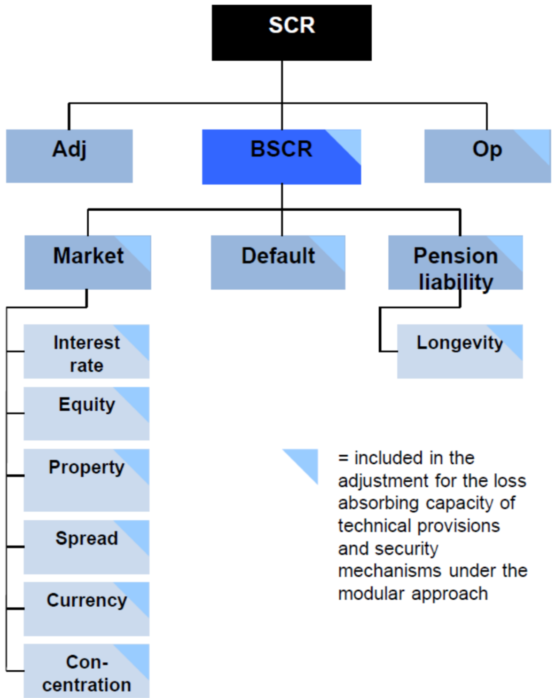

Figure 3: Standard formula modules and submodules.

The BSCR, under this last definition of standard formula[17], instead is compound by the linear aggregation[18] of three components concerning: Market deriving risks (Market risk module), risk deriving from the default of counterparties (Default module), and Pension liability module, which in this formulation of the standard formula concern only the risk deriving from changes in the longevity of members. As clear from the scheme provided above, the Market value module is in turn divided into several submodules, computed by a scenario-based approach with given stress coefficients or a predefined formula

[17]Note, a different definition of the modular structure of the standard formula was given by EIOPA in the 2012 QIS.

[18]Pairwise correlation coefficient provided by EIOPA.

based on balance sheet items.

The introduction of the Holistic balance sheet and a risk-based capital requirement has been openly opposed by numerous member states and, to date, an agreement has not yet been reached for its implementation; indeed, the IORP II final draft, implemented in Italy by D. Lgs. 147/2018, did include only provisions regarding the second and third pillar originally proposed by EIOPA in 2012, excluding so for now, any reference to a risk-based solvency requirement uniform for all member states.

5.2 Risk framework and measures for a pension fund

In this section we recall the settings of a private pension fund, we mark the cash flows out and in, thus defining the various random variables that affect our process. In the second paragraph, we also introduce the concept of risk measure and report some examples of risk measures that will be useful as a benchmark, then in the application part that will be developed in the following chapter.

5.2.1 Notes on stochastic processes concerning pension funds

Based on the model we have described also in the previous sections, we can define the stochastic process concerning the single insured during the adhesion to a pension scheme of the second pillar. It is very important to underline these steps since, given the commitments promised, the financing of the same must necessarily be found endogenously from the assets that are gradually accumulated through the stock of resources and the relative return on investment. To this end, we recall the various states considered by us and not that typically affect the financial status of the fund:

1. Active Worker,

2. Permanent invalid,
3. Early retirement,
4. Retired for old age or seniority,
5. Surviving retired family of active deceased,
6. Surviving retired family of pensioner deceased,
7. Exit the system and delete the position.

Assuming the discrete scheme over time, with annual status surveys, we remind you that the various phases can be differentiated internally. For example, by remaining in the active status, the worker evolves his career with economic effects of salary changes, both attributable to career development, given by promotions or seniority, and to effects attributable to monetary inflation. We also remind you that many steps are irreversible, others impossible, others impossible considering only one year of development.[19]

Taking into account the economic and financial effects regarding the demographic process, bearing in mind both the wages and related contributions from the assets, and the pension benefits for those who have left active status 1 or to surviving families as long as they are entitled to them, the cash flows entering and exiting the fund can be determined in random and expected terms. In particular, we have:

The fund present and future *cash-in*:

1. Contributions, ordinary and extraordinary, payable by specific financiers, mostly also guarantors, such as the employers of a structure relating to dependent work.

[19] See section 4.1.

2. Ordinary contributions, payable by the same members of productive age in order to finance future pension or indemnity benefits in their favor.[20]

3. Interest income on the assets accumulated through the contributions received and any contingencies; however, if over time the assets became a deficit, the interest on the assets would become passive and therefore should be considered in the expenditure side.

The fund present and future *cash-out*:

1. The current benefits for existing pensioners enrolled in the fund, are: invalidity pensions, old age and seniority pensions at the end of the normal working period, early retirement pensions provided for by the fund, pensions to survivors of employees or deceased retirees, any repayments for advances, any refunds to the exits;

2. Similar future benefits for current employees in service.

The dynamic relationship to control the evolution of the *balance of the fund*, year after year, under the assumptions that the cash-flows occurs at the start of the year, can be then schematized as follows:

$$W_{t+1} = W_t\,(1 + j_t) + CI_t\,(1 + j_t) - CO_t\,(1 + j_t) \tag{96}$$

where W_t is the balance at instant t, initial of the period $(t, t+1)$, j_t is the interest rate in force in the period $(t, t+1)$, CI_t are the income for contributions and similar and CO_t relates to all the cash-out flow in the period $(t, t+1)$, supposedly distributed evenly over the period. Operating in terms of expected

[20] In our framework in particular we will relate only to the ordinary contribution, without making an explicit distinction between paid by the employer and paid by the insured member.

values and with a deterministic j_t rate, that is, by determining with actuarial simulation techniques the succession of the average values $E\left(CI_t\right)$ and $E\left(CO_t\right)$ and assigned the *initial resources* W_0, the eq. 6.2.1 allows to determine the sequence of the average values of $W_t, \forall t$. This is a usual valuation of actuaries to highlight, beyond the compilation of technical balance sheets in capital value, the financial dynamics of pension funds and constitutes a first check of the stability of these funds, in the sense that it allows to establish if and when the expected balance it falls below a certain level, for example 0.

Given the multiplicity of decreases considered in our model, the relationship can be further broken down in order to observe the incidence of cash-flows for each type of benefit paid; in this regard we have:

$$W_t = \sum_{i=2}^{5} W_t^i \tag{97}$$

where W_t^i represent the portion of capital related to the decrement i:

$$W_{t+1}^i = W_t^i \left(1 + j_t\right) + CI_t^i \left(1 + j_t\right) - CO_t^i \left(1 + j_t\right) \tag{98}$$

So, the balance of the fund can be expressed in function of the single components, denoted by i:

$$W_{t+1} = \sum_{i=2}^{5} W_t^i \left(1 + j_t\right) + \sum_{i=2}^{5} CI_t^i \left(1 + j_t\right) - \sum_{i=2}^{5} CO_t^i \left(1 + j_t\right) \tag{99}$$

where CI_t^i and CI_t^i are the cash-flows relating to pensions of the state i.

We also report the dynamic relationship to control the evolution of the random variable *Operating result of a pension fund at time t* for a pension fund based on a defined benefit scheme:

$$U_t = C_t \cdot \left(1 + j_t\right) + j_t \cdot F_t - \left[\Theta_t^2 + \Theta_t^3 + \Theta_t^4 + \Theta_t^5\right] - \left\{V_t - V_{t-1}\left(1 + j_t'\right)\right\} \tag{100}$$

where: j_t indicates the interest rate, which varies with time; j'_t represents the rate paid on provisions at the beginning of each year; Θ^i_t, (with $\quad i = 2, 3, 4, 5$) is the value of the charge paid for the departures of active workers due to disability ($i = 2$), old age ($i = 3$) or survivor's pension ($i = 4.5$); C_t represents the collection of premiums capitalized for one year; $F_t \cdot j_t$ represents the profitability of the fund's assets based on the market rate of return of a portfolio of assets; $\{V_t - V_{t-1}(1 + j'_t)\}$ represents the variation in mathematical reserves during the year, also taking into account the profitability of the assets against existing ones.

Assuming a closed group of members, it is possible to additively consider the profit as the union of many funds relative to the same time t, each belonging to a member:

$$U_t = \sum_{k=1}^{n} U_{k,t} = \sum_{k=1}^{n} C_{k,t} \cdot (1 + j_t) + j_t \cdot F_{k,t} - \left[\sum_{i=2}^{5} \Theta^i_{k,t} \right] - \{V_{k,t} - V_{k,t-1}(1 + j'_t)\} \tag{101}$$

where the subscript k indicates that the item is attributable to the specific k-th member of the pension fund.

5.2.2 Risk measures and capital allocation

Formally, a financial risk measure is a function $\rho(.)$ of a risk X such that allows expressing the riskiness of a position with just one number. Obviously, the riskier a position is, the higher its measure of risk will be. When positive, the number $\rho(X)$ assigned by the measure ρ to the risk X will be interpreted as the amount of capital an agent has to add to the risky position X to make it an acceptable position. On the contrary, if $\rho(X) < 0$, the cash amount $-\rho(X)$ can be pulled out from the already being acceptable position and invested more profitably.

However, there are different types of risk measures with different assumptions and different properties. According to the literature has been defined some properties which makes a risk measure to be coherent. A coherent risk measure has specific technical requirements such:

1. *Normalization*: $\rho(0) = 0$, the risk of random variable holding no risk is 0
2. *Monotonicity*: considering two negative random variables X_i, X_j, if $X_i > X_j$ then $\rho(X_i) > \rho(X_j)$; if the values of a negative variable X_i are under almost all scenarios greater than X_j, the risk associated to this last should be lower than the first;
3. *Sub-additivity*: considering two random variables $X_i, X_j, \rho(X_i + X_j) <= \rho(X_i) + \rho(X_j)$ this property in our context implies that diversification is beneficial in terms of risk;
4. Positive homogeneity: considering $t > 0$ and the random variable X, then $\rho(tX) = t\rho(X)$ namely if you double your position in a risky asset, you double also the risk associated;
5. *Translation Invariance*: considering a costant $t > 0$ and the random variable X, then $\rho(X + t) = \rho(X) - t$, for instance imagine to have have a risky asset X and a cash amount t then my risk is covered untill the amount t; if $t = \rho(X)$ then $\rho(X + t) = 0$ (This is the simple principle behind capital requirements defined by different regulations for financial istitutions).

These properties are important since the aim of a risk measure is to evaluate a risk position and so they permit to describe it coherently, according to different factors such as the riskiness of the random variable, to the diversification and dependencies among sub-portfolios in a portfolio.
This topic becomes even more relevant when we use these mathematical tools in order to determine the *Risk adjusted Capital* (RAC) of an economic entity,

which is the amount of capital necessary to cover this risk. This is necessary to have a practical unit measure of risk for purposes as risk-sensitive pricing, portfolio optimization, capital requirements, and related to risk-adjusted return on capital determination. This topic is strictly linked to the so called *cost of capital*, which is the opportunity cost linked to capital allocated to cover the risk. In fact, since the capital requirement is used as a buffer for losses, it cannot be invested in risky asset. But instead it has to be invested in very low risk and liquid assets, which tipically has also low rate of return. In particular business such as the Insurance sector this cost can be a main driver in the economic decision, though a coherent choice of the risk measure can be essential. Though, except for reasons linked directly to the capital allocated, the choice of a risk measure can affect many other risk-driven aspects of a financial institution: they are also extremely important in terms of solvency position, riks appetite, and management risk-driven decisions. Thus, RAC is defined as:

$$K = \rho_\alpha(X)$$

where $\rho()$ is the chosen risk measure and α is the risk tolerance. The choice of the risk measure is fundamental and must be made according to an α level. The α represents the level of confidence we want to assure, in classical risk theory holds the result that to have a higher level of confidence to be covered by risk, we need so a higher amount of capital.

A traditional approach is to face the problem according to a mean-variance efficient frontier, where the best strategy for the time horizon is to maximize the expected value of the capital W_t once fixed the initial capital (equivalent to maximize the return for stockholders) and, at the same time, to minimize its variance. The main shortcoming in using the variance as a risk measure is that according to this view the risk is entailed in all deviations from the mean, without any reference to the algebraic sign. Very often in insurance and finance, the real risk is only the downside risk and then a semi-variance approach would be preferred despite that favorable signs are not counted for in the risk measure.

A well known one-side approach to risk evaluation is the *Value-at-Risk* (VaR) widely used when the risk relies on the occurrence of unfavorable events such as insolvency are to be estimated. It expresses the maximum probable loss at a certain level of statistical confidence $(1-\alpha)$ in a given time horizon:

$$\mathrm{VaR}_{1-\alpha}(X) := \inf_{t\in\mathbf{R}}\{t : \Pr(X \leq t) \geq 1-\alpha\} = F_X^{-1}(1-\alpha) \tag{102}$$

where X is the random variable associated with the risk and $F_X()$ is its cumulative distribution function. Though, have been frequently pointed out that the use of quantile to measures size of the loss does not take properly into account the risk, because no reference is made at the shape of the tail distribution exceeding the quantile Indeed, in two different cases having the same VaR, it may happen to get different expected shortfalls.

Another risk measure which appear to be more *coherent* is *Tail Value at Risk* (TVar):

$$\mathrm{TVaR}_{(1-\alpha)}(X) = E\left[X \mid X \leq F_X^{-1}(1-\alpha)\right] = \frac{1}{\alpha}\int_{-\infty}^{VaR_{(1-\alpha)}(x)} xf(x)dx, \tag{103}$$

It quantifies the expected value of the loss given that an event outside a given probability level has occurred. It basically measures the expected value of the shortfall given a specific confidence level.

Finally, we report the last two risk measures which we will use later in the practical analyses, the *Excess Tail Value at Risk* (xTVaR), in this case, the measure of risk coincide not anymore with the expected shortfall behind the defined quantile, but to the difference between the expected shortfall and the expected value of the random variable underlying the risk:

$$\text{xTVaR}_{1-\alpha}(X) = E[X] - E\left[X \mid X \leq F_X^{-1}(1-\alpha)\right] \tag{104}$$

The *Capital at risk*, denoted by (CaR), given a specific confidence of level and a specific time horizon, measures the risk of unfavorable deviation from the initial capital of the company. It is directly related to the concept of *capital requirement*, in fact, it refers to the amount of capital that must be set aside in order to cover the business from unfavorable deviations of economic variables. In annual terms:

$$\text{CaR}_{\alpha}(X) = X_{t-1} - F_X^{-1}(1-\alpha) \tag{105}$$

In this last two cases, we have a higher level of coherence since the defined risk measure is not just related to the expected shortfall but it also compares the shortfall with the expected value of the specific random variable.

Ruin probability A more heuristic method of measuring risk, which will be useful later to compare different scenarios are the *probabilities of ruin*, or in our case the probabilities of empirical ruin. This measures of risk, unlike the measures seen above, are not necessarily associated with a measure in terms of capital. In this regard they do not give us an exact measure of the economic risk but rather a relative probability of the possibility of failure. For the purpose of the risk analysis we intend to carry out, however, it may be useful to compare quantile risk measures with risk measures in terms of probability such as the ones we will describe below.

We define with *ruin barrier* the capital value below which our fund is deemed bankrupt. The ruin barrier can be the value 0 as any positive value. For example, in the insurance world, thanks to solvency regulations, the limit under which an insurance company cannot continue to carry out its normal activity is defined as a value greater than 0 in order to protect the policyholders. Another example to then measure the risk held could be to set the ruin barrier at a value equal to the initial capital held, so as to be able to determine in terms of probability the risk of financial losses.
Given the initial capital $W_0 = \bar{W}$ and defined the ruin barrier equal to 0, let denote by $p^R_{\bar{W},t}$ the *probability to be in ruin state at year t* irrespective of the ruin or not-ruin state at previous years $(\mathrm{t}-1, \mathrm{t}-2, \ldots)$

$$p^R_{\bar{W},t} = \mathrm{P}\left[W_t < 0 \text{ with } W_0 = \bar{W}\right] \tag{106}$$

On the other hand, the *finite time ruin probability* in the time span $(0, \mathrm{T})$ is the probability to be in ruin state at least in one of the time points $1, 2 \ldots \mathrm{T}$:

$$p^R_{\bar{W},t} = \mathrm{P}\left[W_t < 0 \text{ for at least one } t = 1, 2, \ldots T \text{ with } W_0 = \bar{W}\right] \tag{107}$$

Lastly, the *one-year ruin probability* $p^R_{\bar{W},t-1,t}$ is the probability to fail in a ruin state for the first time at the time point t, having been in a no-ruin state for all the previous years:

$$p^R_{\bar{W},t-1,t} = \mathrm{P}\left[W_t < 0 \text{and } W_h \geq 0 \text{ for } h = 1, 2, \ldots, t-1\right\} \tag{108}$$

Capital Requirement According to the eq. 5.2.2 we can introduce the concept of risk based multi-annual capital requirement:

$$RBC_{\alpha,0,t} = \rho_{\alpha,0,t}(X) \cdot \nu(0,t) \tag{109}$$

given the confidence of level α, the time horizon $(0,t)$, and defined the risk measure, is possible to compute the capital requirement to be held in order to not be in ruin after t years; where $\nu(0,t)$ denote the discount factor for the whole time horizon.

In our application, we want to evaluate the initial capital necessary to not be in default during the whole run-off of the cohort. In this regard, under the *realistic* assumption that the default can occurs only once, we can determine the risk-based capital as the maximum among the expected present values of shortfalls that could occur during the whole lifetime of the cohort:

$$RBC_{\alpha,0,T} = \max\left\{\rho_{\alpha,0,t}(X) \cdot \nu(0,t) \quad \text{with} \quad t = 1, ..., T\right\} \tag{110}$$

However, such a definition of capital requirement is based on strong assumptions, in particular regarding the expected rate of return on capital during the run off of the cohort. In the following chapter we will though consider this relationship in order to see the effect that a possible capital requirement can have on the probability of ruin if introduced as initial capital.

6 Risk Analysis

In the first section of this chapter, we will introduce additional mathematical and financial concepts in order to carry out an inherent risk analysis of a pension fund. In particular, we will then carry out a numerical analysis based on simulations of the pension scheme highlighted in section 4.1. We will gradually introduce higher degrees of stochasticity by introducing the variability of both the inflation rate, used to revalue pensions, and for wage growth, and then we will also analyze the effect on fund equilibrium in the presence of stochasticity in the asset portfolio. However, we will gradually be able to compare the different effects illustrated above on a defined benefit fund and defined contributions, comparing the results relating to risk measures and the probability of ruin.

6.1 Theoretical notions and model setup

In order to perform the analyses described above we have divided the computations into three principal components, the *states simulation*, the *financial simulation* and the computation of results, for what concern the third part we have already seen in chapters 4 and 5 all the notions necessary to interpret the results and to use raw data to define the different objective statistics.
For the simulation part, and financial simulation we list below the setup of our model.

6.1.1 States simulation process

For what concerns the multi-state transitions, according to the model described in section 4.1, to project for future years the status of the fund we have simulated, according to *multinomial process*, the stationarity/transition of each member at each future years. This has been possible thanks to the Markov property delineated in section 3.3.1. We have indeed assumed each transition independent from the previous passages.
The simulation of transitions over time has been made singularly for each member of the fund, in order to keep information about salary development, seniority and accrued benefit. For this reason we can relate to the *member history* as a stochastic process of random variables distributed according to a *multinoulli distribution*[21].

[21] Called also *Categorical distribution*.

We name $S_x(t) = k$ with $k = 1,2,3,4,5$ the random variable indicating the state, the member aged x, is in at time t. Given that $S_x(t)$ is distributed according a categorical distribution with probability mass function $f(s = k \mid \boldsymbol{p}) = p_k$[22] and support $s \in \{1,2,3,4,5\}$ where $\boldsymbol{p} = (p_1, \ldots, p_5)$, p_i represents the probability of seeing element i and $\sum_{i=1}^{k} p_i = 1$.
Then the *member history* is defined by the stochastic markovian process $\{S_x(t)\}_{t\in[0,\omega-x)}$.

We need further to remark that the sum of mutually independent Multinoulli random variables is a *Multinomial distribution*, so the overall process of the portfolio can be seen as a *Multinomial stochastic process.*

In particular to define the future paths made by an insured member of age x at enrollment date ($t = 0$), we have made recursive simulation from multinoulli distribution using as transition probabilities at each step the set $\boldsymbol{p_t} = \left(p_{x+t}^{1,1}, q_{x+t}^{1,2}, q_{x+t}^{1,3}\right)$, if the starting state was *active* (state 1) or $\boldsymbol{p_t} = \left(p_{x+t}^{2,2}, q_{x+t}^{2,3}\right)$ if the starting state at time t were *disability pension* (state 2). For what concern a member in state *age retirement*, we have basically assumed as transition probabilities the set $\boldsymbol{p_t} = \left(p_{x+t}^{1,1} + q_{x+t}^{1,2}, q_{x+t}^{1,3}\right)$. In general, once the death of the member has been drawn, a further *Bernoulli*[23] draw has been made in order to simulate the presence of survivors using the probabilities to leave a family (Θ_{x+t}^{f}) provided by the *Technical specifications of 22/06/2015* relative to the review of Transformation coefficients of 2016. From the same technical an-

[22] Another formulation of the PMF of a categorical distribution is:

$$f(x \mid \boldsymbol{p}) = \prod_{i=1}^{k} p_i^{[x=i]}$$

where $[x = i]$ evaluates to 1 if $x = i, 0$ otherwise.

[23] Bernoulli distribution can be seen as a specific case of the mulinoulli distribution where the support is $k \in \{0,1\}$ and the probabilituy mass function can be defined as:

$$f(k,p) = \begin{cases} q = 1 - p & \text{if } k = 0 \\ p & \text{if } k = 1 \end{cases}$$

or $f(k,p) = p^k(1-p)^{1-k}$

nex, we have also obtained the probabilities for a survivor to remain entitled to receive the pension. These probabilities were used to simulate the remanence in states 4 and 5 or the eventual transition to state 0 (end of contract).[24]

Mortality scenarios: For what concerns the probabilities, the technical bases provided by the book ”Assicurazioni sulla salute, P. De Angelis, L. Di Falco” were used to price and simulate the transitions between the states of active, disabled, and death. We need to point out that those technical bases have been intended to price Long term care insurance contracts, therefore they are not tailored for the field we dealing with. Though the probabilities provided are divided into 3 scenarios of mortality, high, medium, and low, this will be useful later in order to make some sensitivity analyses. We show in the next graphs respectively the survival function, death function and disability function of the three scenarios mentioned above for male members with age $x = [20, 125]$:

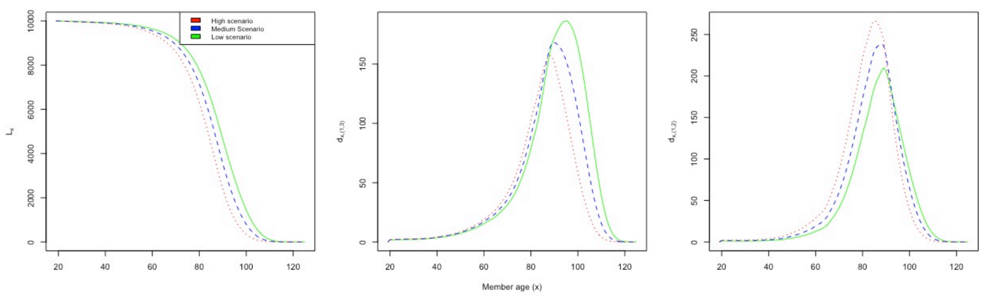

Figure 4: In order: survival function, death function and disability function for a cohort of male members.

In red we have the scenario with a higher death probability while in green

[24]The estimation of probabilities to leave a family and to keep the pension rights by widow/widower has been made by ISTAT in 2013 for the review of Dini coefficients in 2016.

we see the scenario with a higher survival probability. As we can see as the probability of death at age x increases we have a shift in both the probability to become disable and to remain active. The increase of death probability in the different scenarios will have some effect obviously on both the number of people in state 2 and state 1 at each future year, with some contrasting effect on what concerns the total amount of benefit to pay to future pensioners and relative pricing.
A similar behavior holds for what concern the probability to stay in state 2 over the years and the transition probability between state 2 and 3:

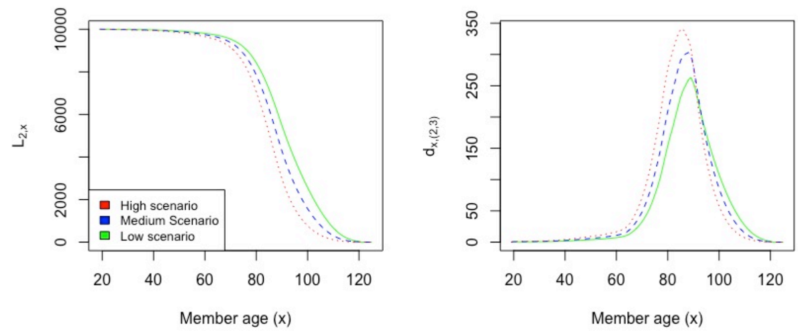

Figure 5: Survival function of disability pension receiver and relative death function of female members.

To complete what has been said so far, two brief simulations (1000 subjects for 1000 iteration) of the stochastic process that governs the number of members of a cohort of entrants aged 25, within each state during the years of projection, is illustrated below. The first set of graph concern the low mortality scenario while the second the high mortality scenario. In particular for each scenario, we have analyzed the trend of the number of the various states for homogeneous subjects, males aged 25 at entry. To this end, the graphs illustrating are divided into 3 parts. First, we observe the development of state 2 presences relating to the disability pension receptors, in the second we have the number of subjects who enter and remain in state 4 while in the third graph we have the contracts relating to survivors of retirees. Remember that, as seen in paragraph 4.1, a differential $c = 3$ years was used between the assignor and the widower/widow to project the stay in states 4 and 5. We further remember that according to our setup the entry in state 2 is stopped starting from the reach of the retirement age (which occurs at the $t = 43$ since the entry age of the cohort is 25).

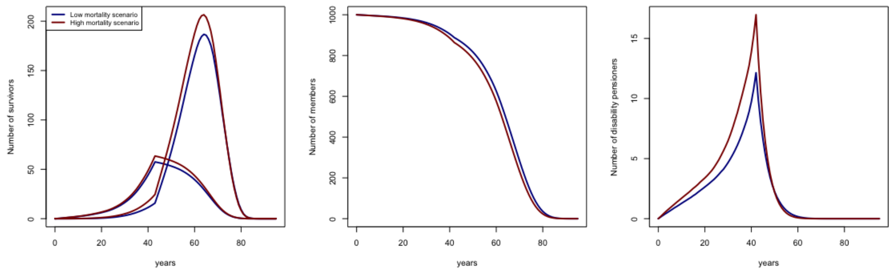

Figure 6: Expected number of people during the whole runoff of a cohort of entrants aged 25. Comparison between scenarios, result of 10000 simulations.

To compare the results we report a table with distribution statistics at projected year 25, 50 relative to the different states[25]:
We report for completeness the results relative to the first three moments of the distribution of the number of people at time 25 and 50. We remark that the starting age of the cohort is 25, so in the first part the basic scenario concerns active workers while the second concerns retired pensioners. The modest number of simulation does not permits to have sufficient accurate statistics for what concern the third and fourth moments. Though we can notice a consistent pattern at both ages. Firstly we denote the expected number of people in

[25]Since the passage from state 1 to 3 is bounded at the age 68 in the table we have merged the results, the third row instead show the cumulative result of survivors (state 4 and 5).

State	Mortality Scenario	*Mean*	*Sd*	*Skewness*	*Mean*	*Sd*	*Skewness*
		$t = 25$			$t = 50$		
1	H	977.98	4.72	-0.27	855.22	11.11	-0.04
	L	985.68	3.75	-0.12	908.32	9.04	-0.01
2	H	6.63	2.55	0.32	0.89	0.98	1.22
	L	3.59	1.91	0.43	1.29	1.15	0.91
4+5	H	8.24	2.99	0.37	96.87	9.13	-0.04
	L	5.62	2.29	0.35	60.17	7.22	0.09

Table 2: Characteristics of the distribution relative to the number of people, comparison of the results for the years 25 and 50 and for different mortality scenarios.

the various state, in fact in the High mortality scenario we see a lower number of people in state 1 ($t = 25$) and state 3 ($t = 50$), this is obvious since the permanence in those states is bounded to the mortality or disability rates. For what concerns the number of survivors, which, given constant the probability to leave a family, depends only on the mortality rates at a given age, we can see that the behavior is as expected opposite to the one of the states 1 and 3. For what concern instead of the average people inside the group 2 at a given time among the different scenario, the behavior is opposite for different times. We have to remark though that the possibility to go in the second state is stopped after the reach of age 68. In fact, at time $t = 25$ we have a higher presence in the disability state of the high scenario with respect to the low scenario since the disability rate is more several (as seen also in figure 4) at an early age, while at $t = 50$ the higher mortality rate of the high scenario has a higher impact, giving so a lower number than the low scenario.

As regards the standard deviation, on the other hand, we can observe consistently at both times analyzed, that it is higher in the high scenario than in

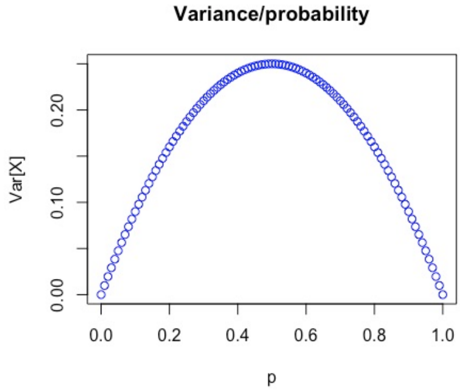

Figure 7: Relation between the parameter (representing the probability of success) and variance in a Bernoulli distribution.

the low scenario, regardless of the state we are talking about. This is due to the theory behind our model as it is easy to deduce in a multinomial process the highest variance is when the probabilities defining the probability mass function are uniform. This concept is also verifiable simply in the specification of a Bernoulli experiment. In this regard, the variance can be determined by $\text{Var}[X] = p(1-p)$, and therefore the more the probability approaches 50% (iso-distribution), the higher the variance. In this regard, we can justify this phenomenon with the probabilities of transition from one state to another. While in the low scenario we will have a very high probability of surviving and low probabilities of becoming disabled or dying, in the high scenario these probabilities will be (slightly) more uniform, indeed the probability of survival is lower while the probabilities of transition are higher.

The incidence of the number of enrolled: Finally, we analyze the incidence of the number of cohorts on the statistics relative to the distribution of our members over time.

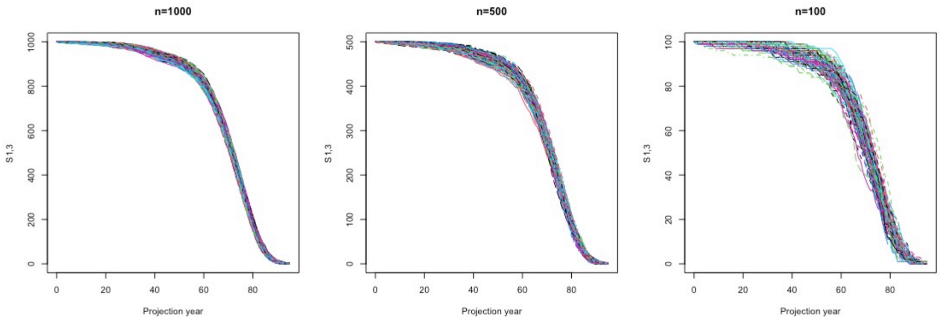

Figure 8: Simulation paths for active members to retired pensioners, for cohorts of 1000,500,100 male insured with age at the entrance of 25.

In this last test, we go back to our cohort of males aged 25 upon entering the fund and analyze what happens in terms of distribution as the number n of this cohort of entrants varies. In the first case, we have 1000 members, in the second case 500 and the third 100. To this end, we have screened the process 1000 times for each insured person for 100 years. As can be seen from figure 8, the trend of the population is very similar, however, it is possible to detect, in relative terms, greater volatility for a lower number of members at the entrance. This effect naturally arises from risk diversification, for which a larger fund naturally manages to *control* volatility. We observe in the following set of graphs the distributions of the simulation results for state 1 and state 2 at time $t = 30$. It is possible to find a decreasing trend of the *Coefficient of variation* (CV)[26] as the number increases. In particular, from the table above, we see how in both States, while the mean and the standard deviation increases with the number of people increases, we have a lower incidence of the standard deviation on the mean in the large cohort to other cohorts.

[26] The Coefficient of variation is a measure of relative volatility, defined as:

$$CV(\mathbf{X}) = \frac{\sigma(\mathbf{X})}{E(\mathbf{X})}$$

$t = 30$	State 1			State 2		
n	1000	500	100	1000	500	100
$Mean$	978.05	488.87	97.87	4.552	2.25	0.42
Sd	4.594	3.138	1.418	2.097	1.511	0.606
CV	0.46%	0.64%	1.44%	4.60%	6.71%	14.42%

Table 3: Summary statistics of distribution of the variable *number of people* relative to the State 1 and State 2 for the year 30.

Another effect that is attributable to the increase in numbers is the progressive decrease of the skewness (see figure 9) in the distribution of the number of members. In this case, the effect is attributable to the *central limit theorem*, so that our distribution as n increases therefore tends to a Gaussian distribution.

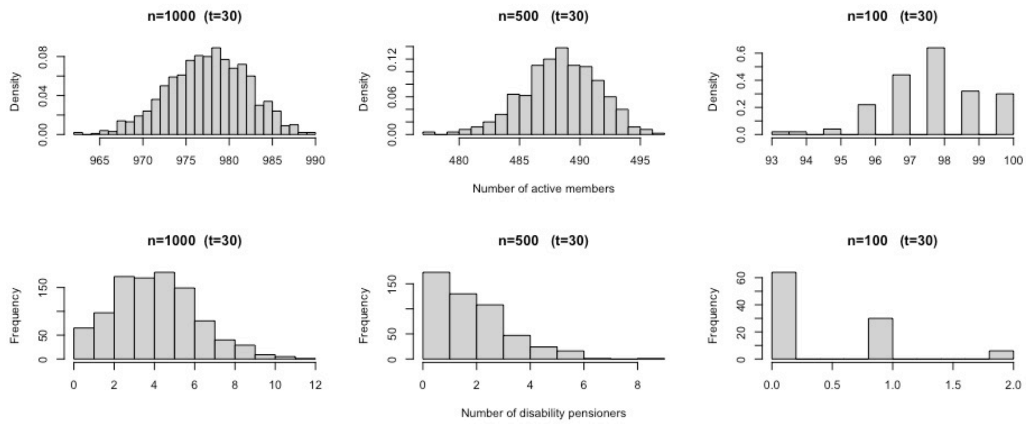

Figure 9: The figure shows the distribution of the variable *number of people* at the projected year $t = 30$, for different states and number of members in the cohort.

6.1.2 Financial simulation, inflation rates and return on investments

So far we have seen what concerns the simulation of the number of people in the various states for the future years to enter the pension system. However, in order to have a complete view of what concerns the variability and solvency of a pension fund, it is also necessary to take into account the aspects concerning the revaluation of pensions and the profitability of assets. To this end, we will use stochastic processes used to evaluate financial instruments to simulate, under realistic assumptions, the returns on assets accumulated by the pension fund and the future inflation then used to annually reassess the contributions to be paid to pensioners. For completeness and a better understanding of the topics explained and that we will explain, we report below the characteristics of the Wiener process and the Euler-Maruyama method for approximation of stochastic differential equation.

The characteristics of a Wiener process B_t (sometimes called also Brownian motion) are:

1. $B_0 = 0$
2. B has independent increments: for every $t > 0$, the future increments $B_{t+u} - B_t, u \geq 0$, are independent of the past values $B_s, s \leq t$
3. B has Gaussian increments: $B_{t+u} - B_t$ is normally distributed with mean 0 and variance u: $B_{t+u} - B_t \sim \mathcal{N}(0, u)$
4. B has continuous paths: B_t is continuous in t.

Some results follow immediately after the definition of Wiener process properties:

- Since the process has independent increments means, if $0 \leq s_1 < t_1 \leq s_2 < t_2$ then $B_{t_1} - B_{s_1}$ and $B_{t_2} - B_{s_2}$ are independent random variables, and the similar condition holds for n increments.

- The unconditional probability density function, follows a normal distribution with mean $= 0$ and variance $= t$, at a fixed time t:

$$f_{B_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$$

- Since the increments have a normal distribution: $B_t = B_t - W_0 \sim N(0, t)$
- The expected value of B_t is 0: $E[B_t] = 0$
- The variance of B_t is t: $\operatorname{Var}(B_t) = t$

For what concerns instead the Euler–Maruyama method, it is a method for the approximate numerical solution of a stochastic differential equation. According to this method we consider the SDE:

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}B_t \quad \text{with} \quad X_0 = x_o$$

where B_t stands for the Wiener process.
The approximation to the true solution X in the interval $[0, T]$ is the Markov chain Y defined as follow:

1. Split the interval $[0, T]$ in N equal subintervals of width $\Delta t > 0$ such that:
   $0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$ and $\Delta t = T/N$,
2. Set $Y_0 = x_0$,
3. Given $\Delta B_n = B_{\tau_{n+1}} - B_{\tau_n}$ we recursively compute Y_n for $1 \leq n \leq N$ as:

$$Y_{n+1} = Y_n + a(Y_n)\,\Delta t + b(Y_n)\,\Delta B_n$$ [27]

[27] As seen previously $\Delta W_n \ \forall\ 0 < n < N$ are indipendent and identically distributed according to a gaussian distribution with $E(\Delta B_n) = 0$ and $Var(\Delta B_n) = \Delta t$.

Inflation rate: In order to generate future inflationary scenarios we have used the *Ornstein-Uhlenbeck*[28] stochastic process defined as:

$$di_t = k\,(\theta - i_t)\,dt + \sigma dB_t \tag{111}$$

This process is a *mean reverting process* since the level of inflation tends toward the average level denoted by θ. The choice of an average reversal process has foundations in European monetary policy, in fact, it is consistent with the inflationary objective pursued by the ECB since its establishment. In this regard, it is recalled that the first objective of the European central bank is price stability and that the target is to maintain the annual variation of the harmonized index of consumer prices, lower and close to 2
Going back to the formulation, it can be divided into two parts, the first part on the right-hand side of the equation is the drift and indicates the expected change in the level of inflation over the next interval of time. The second part instead is the diffusion, which represents the stochastic component of the process. q_t is the level of inflation at time t, k instead is the speed reversion parameter, which indicates how quickly the rate of inflation reverts to its mean. Without the diffusion component, the process would converge to the mean, though the term B_t, which represents a *Wiener process*[29], together with the parameter σ insert uncertainty in the process. In particular the parameter k and the parameter σ set which part of the equation has the most impact. If σ is large, the uncertainty exhibited by the Wiener process is magnified and any reversion toward the mean inflation rate may be overshadowed by the diffusion process. However, if k increases, relative to σ, then mean reversion dominates the movement of inflation.
By the way, rather than a continuous description of the stochastic process, we are interested in generating *annual* inflation rate. In this regard we have used

[28] In finance known also as *Vasicek model.*

[29] See 6.1.2

the *Euler–Maruyama method* seen above in order to discretize the above process and thus obtain a recursive discrete definition to be used for our purposes. According to this, the Euler approximation of the Ornstein-Uhlenbeck process is:

$$i_{t+1} = i_t + di_t = i_t + k(\theta - i_t)\Delta t + \sigma \Delta B_t \tag{112}$$

using *annual steps* we have $\Delta t = 1$ and so the discrete time equation is an autoregressive time series model of first order:

$$i_{t+1} = i_t + di_t = [(1-k)i_t + k\theta] + \sigma\varepsilon_t \tag{113}$$

where ε_t is a draw from a standard normal distribution (see footnote 3). The term between square brackets shows that the expected future inflation is an average of the current level of inflation i_t and the long term mean θ. The parameter k is the weight which regulate the weighted average. Note that if $k > 1$ and the current level of inflation is below its mean, next year's inflation is expected to exceed the mean.[30]

Concerning the variability of the process, the volatility of an autoregressive process increases with the projection periods. In particular, we can demonstrate that, with annual time steps, the volatility of our autoregressive process tends to an equilibrium level $\bar{\sigma}$:

$$\text{Volatility } i_t \rightarrow \frac{\sigma}{\sqrt{1-(1-k)^2}} = \bar{\sigma} \tag{114}$$

Though the volatility in the long period does not depends only on the *sigma*

[30]In case of higher number of steps (for instance monthly) we have:

$$i_{t+\Delta t} = \{[1 - k(\Delta t)]i_t + k(\Delta t)\theta\} + (\sigma\sqrt{\Delta t})\varepsilon_t$$

parameter associated with the diffusion component.

In particular in the case that k is bounded between $(0, 2)$ the process volatility converges to the value seen in eq. 114[31], while if out of bounds it diverges.

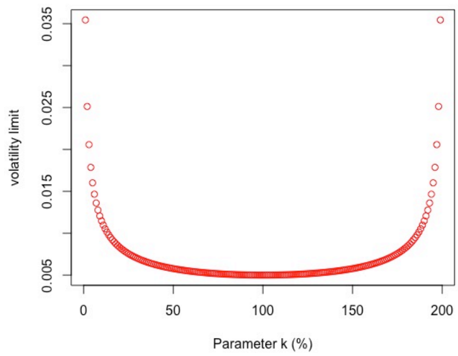

Figure 10: Grafical representation of $\sigma(i_t)$ with $t \to \infty$ with $k \in (0, 2)$

We have further demonstrated numerically that in the case k is equal to the bounds discussed above, the variability of the process grows according to $\sqrt{t}$:

t	1	10	30	50	70	100
$k \simeq 0$	0.0050	0.0157	0.0273	0.0353	0.0418	0.0500
$k = 1$	0.0049	0.0049	0.0050	0.0049	0.0049	0.0049
$k \simeq 2$	0.0050	0.0158	0.0274	0.0353	0.0416	0.0499
$\sigma\sqrt{t}$	0.005	0.0158	0.0273	0.0353	0.0418	0.05

Table 4: Results concerning 1'000'000 simulations, with $\sigma = 0.5\%$, $\theta = 2\%$ and $i_0 = 2\%$.

In fact, we have that, when the mean reversion speed (k) is high (for instance 0.9), the process has memory loss and as the projection period grows, volatility is tied to the current environment. However, when k is low (for instance 0.01)

[31] In case $k = 1$, the equilibrium level is $\bar{\sigma} = \sigma$

the inflation process is less tethered to an average inflation level and uncertainty grows. This effect can be showed clearly in the picture above, where

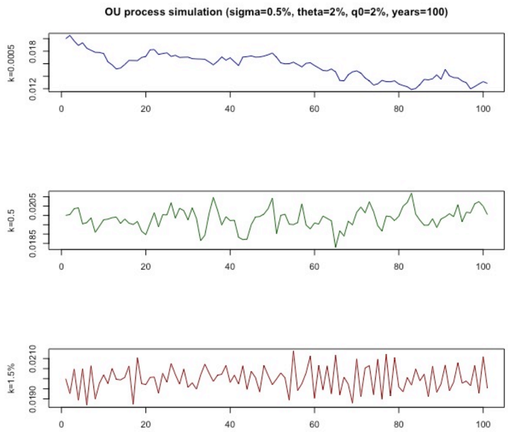

Figure 11: Inflation projection with different level of k.

we have plot for 100 periods processes with different level of mean reversion (respectively $k = 0.005,\ 0.5,\ 1.5$), the other parameters are keep constant in order to evidence the effects of mean reversion change.

Finally we have stressed the σ parameter, keeping fixed the rest.

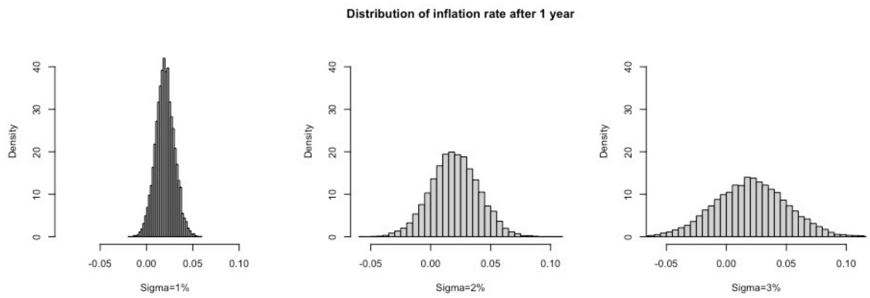

Figure 12: Empirical distribution relative to a 1 year projection with different parameter σ ($k = 1\%$, $i_0 = 2\%$, $\theta = 2\%$).

As expectable the change of σ affect directly the volatility of the process since it increases the incidence of the diffusion component which is the stochastic part of the process. We have simulated in figure 12, one year projection for different level of sigma, the increase of σ, as can be seen from the figure, has a strong effect on the variability, despite no effect in terms of expected value. A stress on this parameter would be useful later where we will set different scenarios in order to test the resiliency of pension funds balance to inflation rates stochasticity.

Stochastic financial returns As regards the simulation of the so-called market risk, for simplicity we have decided to define a portfolio composition ex-ante. In particular, the β portion will be invested in *low-risk assets* while a portion $(1-\beta)$ will be invested in *high-risk assets*. The rate of return obtained will then be the weighted average of these two rates. We are aware of the fact that this method is simplistic and does not consider some factors such as risk concentration, counterparty risk, spread risk, and Asset-Liability frameworks. However, here we are mostly interested in giving a taste of how the volatility of the rate of return affects the balance of the pension fund and how it can have a strong effect on the capital allocated to hedge the risk.
As regards the models for the rates of return of instruments whose risk profile is low, such low risk corporate and government bond we have used the same criterion seen to model the inflation: The *Vasicek* differential equation:

$$dr_t = \alpha(\gamma - r_t)dt + \sigma dB_T \tag{115}$$

The use of a *mean reversing process* in order to model the low risk interest rate is justified by the theory. In particular when rates are high, the economy tends to slow down and there is less demand for funding. Rates therefore decrease. Conversely, when rates are low, there tends to be a greater demand for funding and this leads to an increase in rates. By using the Ornstein-Uhlenbeck process, we consequently assume the possibility of generating negative interest rates, a

scenario that is not entirely impossible considering the yields of the government bond market in recent years.
Applying the Euler-Maruyama method we have:

$$r_{t+1} = r_t + dr_t = r_t + \alpha\left(\gamma - r_t\right)\Delta t + \sigma \Delta B_t \tag{116}$$

and with annual steps:

$$\begin{aligned} r_{t+1} &= r_t + \alpha\left(\gamma - r_t\right) + \sigma\varepsilon_t \\ &= r_t - \alpha r_t + \alpha\gamma + \sigma\varepsilon_t \\ &= \left[\left(1-\alpha\right) r_t + \alpha\gamma\right] + \sigma\varepsilon_t \end{aligned} \tag{117}$$

where ε_t is still a draw from a standard normal distribution and the consideration made above holds.

Finally, the usual scheme seen above was used to generate the rate of return for the portion of the portfolio invested in high-risk instruments such as equities. To this end we used as reference the *Black-Scholes* stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{118}$$

where B_t still represents the standard Wiener process, the initial value $S_0 = s_0$ is fixed and $\sigma > 0$. The drift parameter μ is a time-constant interest rate, which coincides with the expected annual rate of return in S_t and σ, the diffusion parameter, represents the annual volatility of a risky activity. The first assumption is that the instant rate of change is distributed according to a normal distribution with mean μ and standard deviation σ. Therefore in the interval Δt the rate of change of the price of a share will have mean $\mu\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$. In formulas, from equation above we have:

$$\frac{\Delta S}{S} \sim N\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

Where ΔS is the change in the stock price in the interval Δt and $\sigma^2 \Delta t$ indicates the variance of the normal distribution. Defined, S_T the price at time T of the equity, and S_0 the equity's price at time =, from the Ito's Lemma we have that:

$$\ln\left(S_T\right) - \ln\left(S_0\right) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right]$$

and so,

$$\ln\left(S_T\right) \sim N\left[\ln\left(S_0\right) + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right]$$

Since $\ln S_T$ follows a Gaussian distribution we have that the price of the equity S_T follows a logNormal distribution. Given that the first moment of the log-Normal distribution is $e^{\bar{\mu}+\frac{\bar{\sigma}^2}{2}}$ and its variance is defined by $e^{2\bar{\mu}+\bar{\sigma}^2}\left(e^{\bar{\sigma}^2}-1\right)$[32] we have:

$$E(S_T) = e^{\bar{\mu}+\frac{\bar{\sigma}^2}{2}} = e^{\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T + \frac{\sigma^2}{2}T} = S_0 e^{\mu T}$$

and,

$$\begin{aligned}
\mathrm{VAR}\left(S_T\right) &= e^{2\bar{\mu}+\bar{\sigma}^2} \cdot \left(e^{\bar{\sigma}^2} - 1\right) \\
&= e^{2\ln S_0 + (\mu - \frac{\sigma^2}{2}) + \sigma^2 T} \cdot \left(e^{\sigma^2 T} - 1\right) \\
&= S_0^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right)
\end{aligned}$$

This results are useful in order to evaluate the distribution characteristics of the rate of return of equities during the period $(0, T)$. In particular we have that, denoting as $\rho_{0,T}$ the annual rate of return on the period $(0, T)$ we have that:

$$\rho_{0,T} = \frac{1}{T} \ln \frac{S_T}{S_0} \tag{119}$$

[32]where $\bar{\mu}$ and $\bar{\sigma}^2$ are the lognormal parameters.

and so,

$$\rho_{0,T} \sim N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right)$$

Going back to eq.118, according to the Euler approximation method seen above, we have:

$$S_{t+1} = S_t + dS_t = S_t + \mu S_t \Delta t + \sigma S_t \Delta t \Delta B_t \tag{120}$$

and with annual time steps:

$$\begin{aligned} S_{t+1} &= S_t + \mu S_t + \sigma S_t \varepsilon_t \\ &= S_t (1 + \mu) + \sigma S_t \varepsilon_t \end{aligned} \tag{121}$$

With ε_t is a draw from a standard normal distribution. In this case, the process is not mean reversing but instead it grows according to the parameter μ. For what concern instead the volatility, as we can notice, we have a level effect; though here the level of S_t has a strong influence on the diffusion component and so on the overall volatility of the process. In fact, according to Black-Scholes SDE, the σ parameter is weighted for the value of the random variable S_t.

Given j_t the *overall return on portfolio* we have:

$$j_t = \beta r_t + (1 - \beta)\,\rho_{t-1,t} \tag{122}$$

and so the overall return of our portfolio at time t, under our assumption, can be seen as the weighted average of the two processes.

The model could be further complicated by introducing the dependency among shocks in the bond market and equity market or also dependency between the financial sector and economic scenarios. The latter is difficult to implement due to problems in estimating the covariance between the inflationary trend and the two financial sectors mentioned above, in particular during the last 50 years the dependence between *equity market* and *inflation rate* has changed considerably over the years and in a different way depending on the country considered. In this regard, an adequate analysis is outside the scope of this work.

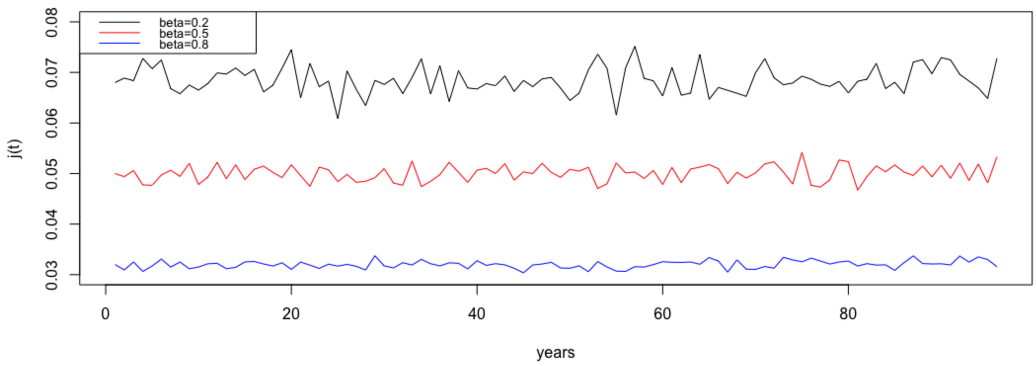

Figure 13: Average return per year of 1000 simulations with different composition portfolio ($\sigma_{hr} = 0.08$, $\sigma_{lr} = 0.005$).

So we assume for now the independence between the two financial sectors. From the next image we can see the difference in terms of expected returns for the projection of portfolios with different composition between high-risk and low-risk assets. We set the expected return of high risk assets at 8% while those at low risk 3%, with initial values equal to the expected values. We remark that β denote the portion of asset invested in low-risk assets, according to this we can see how a higher β corresponds both a lower rate of return but also lower volatility. The graph above shows the average return of 1000 simulations so the extreme values are smoothed by the different simulations, despite this the higher volatility is evident in a high-risk asset allocation.

6.2 Practical Analysis

In this section, we will aggregate all the information discussed above in order to make some practical analysis of the pension schemes we have hypothesized and discussed in the previous section. In particular, we will divide the analysis into several parts. We will start from more standard hypotheses and gradually we will introduce more levels of stochasticity, thus analyzing the difference from the previous steps. Within the same levels of stochasticity, we will vary parameters and assumptions to highlight critical points of our analysis. In almost all of the scenarios, we simulated the entire reference cohort 1000 times. We know that 1000 times may not always be a sufficient number to accurately identify all the characteristics of the distribution, however, we want to highlight (considering only the multi-state model) that 1000 simulations of a cohort of 1000 people per 100 years are equal to a number of simulations of 1 billion. In this regard, the computation time made it difficult to carry out a high number of simulations in each scenario we hypothesized.

6.2.1 Static inflation and return rate

We start by computing, according to what we have seen in the chapter dedicated to the Multi-state model for pension fund (section 3), the expected present value of salaries and benefits for a DB pension scheme, with salary function:

$$f(w) = \frac{\sum w_t}{T} \text{ with } t = 1, 2, \ldots, T \tag{123}$$

where T is the total seniority of a worker, and the function represents the average salary of the career. According to this we have computed the *Cohort Premium* and the relative *Mathematical reserve* of the base scenario, characterized by this parameter:

Where k (lowercase) is the return rate recognized to the member. In particular a $k = 150$ correspond to a *replacement rate* with 40 years of seniority of

k	β	ω	K	γ	j	i	s	MD Table	c
150	68	120	60%	1	5%	2%	5%	Medium Scenario	3

Table 5: Parameters used to price and simulate the defined benefit pension scheme.

$\frac{40}{k} \simeq 0.26$. The percentage of pension given to survivors is denoted by K (uppercase) and is fixed to 60%. The old-age retirement is fixed at $\beta = 68$ and the limit age of the multiple decrement table is 120. We have decided to not set any penalization in case of disability retirement since the lower seniority penalizes already the amount of pension given, so γ has been set to 1.We have set a return on investment rate j equal to 5%, an inflation rate i equal to 2%, and a salary growth s equal to 5%.

About the latter: we can see the parameter s as the sum of two components, inflation, and career growth, by setting it at 5% we have implicitly set the growth of the salary due to career growth at 3%.

The overall wage growth rate has been assumed to be 5% since a mid-career subject with such a growth rate will have a salary in line with or a little higher than the average Italian salary.

In the table below [33] we can see the head and the tail of the table with the expected value of our salaries, benefits (total and divided for decrement) and contributions of our pension scheme.

As we can see by table above and the following images, the amount of salaries grows during the time, this is because in the early years the effect of salary growth is larger than the people becoming pensioners. The same behavior of growth can be seen also in the benefits arising by disability or survivors, though

[33] B represents the total amount of benefits, $B2$ the benefits related to disability salary, $B3$ are the amount of benefits related to the old-age retirement, $B4$ and $B5$ represent respectively the benefits for survivors of workers and pensioners.

	S	B2	B3	B4	B5	B	P	C	V
0	$12,000,000$	0	0	0	0	0	0.0523	$627,600$	0
1	$12,571,716$	19	0	11	0	29	0.0523	$657,678$	$664,431$
2	$13,066,511$	52	0	31	0	83	0.0523	$683,563$	$1,388,183$
3	$13,579,088$	122	0	66	1	190	0.0523	$710,378$	$2,175,246$
...	...	...	...	...	...	...	...	...	...
91	0	0	360	0	0	360	0	0	404
92	0	0	46	0	0	46	0	0	46
93	0	0	0	0	0	0	0	0	0

Table 6: Projected amount, B represents the expected value of benefits, S the expected value of Salaries and C those of contributions, V and P are respectively the mathematical reserve at year t and the premium calculated and then used to compute the projected contributions.

the main effect is given by the people leaving the status 1. In the bottom part of the table, we see how some pensioners are still alive, at extreme ages though no people of the cohort is paying any contributions since at year 43 all the remaining people in status 1 leaves.

The difference in timing between pensions and the collection of contributions can be seen by the following picture:

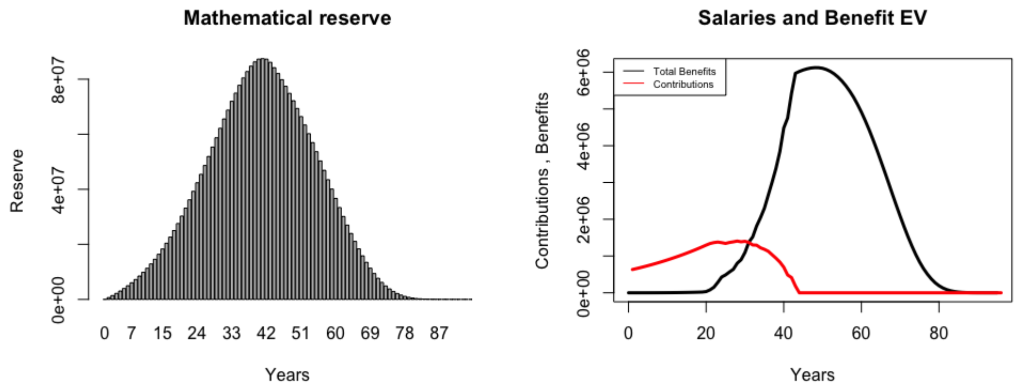

Figure 14: Mathematical reserve and Expected value of benefits and contributions collected by the fund over the years.

In particular, we can see from the picture the magnitude of the total benefits on the amount of contributions collected, what cover this gap is the time value of the money. Thanks to the mathematical reserve (figure 14), since the contributions are collected in early years, they, together with the return accrued during the years are able to cover the amount of benefits arising from the pension scheme.

The analysis can be further extended by considering the single participation of the states to the total benefits. In particular, as expectable the State 2 and State 4 benefits typically occur at an early age with respect to the others. In fact, retirement pension is bound to the reach of the age 68, while the state 5 concerns a two state transition.

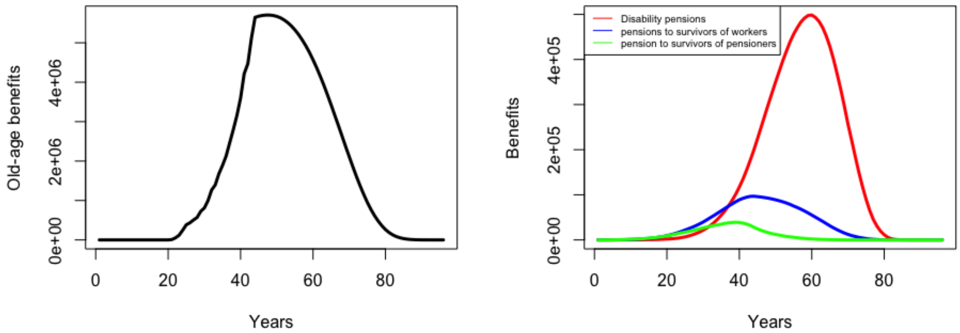

Figure 15: On the left we have the benefits for old-age retirement, on the right the other type of benefits amount.

Though from figure 17 is clear the magnitude of old-age pension with respect to other types of pensions. This is due both to a higher likelihood to reach alive the retirement age and higher amount of pension per capita, since, according to our assumption, to higher seniority correspond also a higher replacement rate.

In order to go deeper into the analysis, we have calculated the reserve dedicated to each type of benefit and the related premiums:

	S	B	B2	B3	B4	B5
EPV	342, 427, 717	17, 913, 792	176, 572	16, 420, 404	388, 915	927, 899
Premiums		0.05231	0.00051	0.04795	0.00113	0.00270

Table 7: In the firs row: net present values at time 0, of salaries and benefits. In the second row, premiums related to different type of benefits.

From the table 15, we can find, that the largest component of the premium corresponds to that covering old-age pensions. The portion of premium rate covering the state 3 pension is 4.7%, while the sum of the other components is only 0.4%, with a total pure premium rate of 5.2%. In particular, it is possible to highlight that this is because the present value of the benefits to be paid for old age retirement corresponds to 91.6% of the total benefits EPV. We now turn to the results regarding our baseline scenario with constant inflation and return rates over time. In this regard, we report the *Balance of the fund* relationship seen in subsection 5.2.1.

$$W_{t+1} = W_t\,(1 + j_t) + CI_t\,(1 + j_t) - CO_t\,(1 + j_t)$$

Through this relationship it is possible to calculate the balance of the fund for each future year for each simulation, so as to be able to construct confidence intervals and the relative distributions for each period.

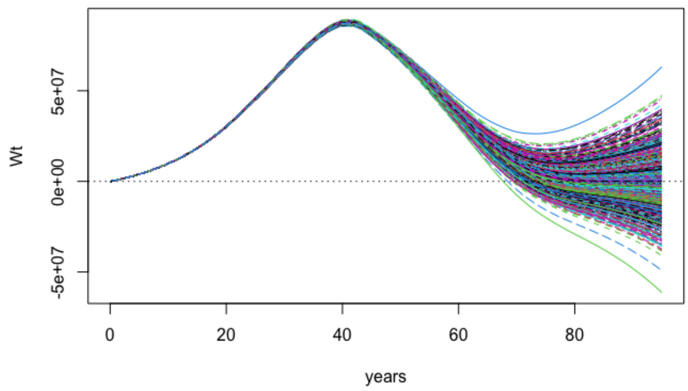

Figure 16: Projection paths of Wt, base DB scenario, $W_0 = 0$

In this first iteration we have taken an initial capital equal to 0 in order to highlight the unfavorable deviation impact on the solvency of the fund. In order to evaluate the magnitude of shortfalls, in our projection we have assumed that the fund continues to operate even in the event of negative capital. For this purpose we assume that the money needed to pay the benefits, in the event of negative capital, is borrowed and the interest rate on that loan is equal to the rate of return on assets.

From the projection we can clearly observe an increase of the curve in the first part, since in that time a higher amount of contributions is collected. Contributions which will then used, together with returns on capital to pay the benefits arising by the pension of members. At the run-off of the cohort contracts the expected remaining capital is equal to 0. The expected balance of the fund converge to 0 since the assumptions used to price the contributions paid by members, are the same used to project the balance of the fund.[34]. For what concern the variability, we have a strong increase of it at later years, this is mainly due to the increasing volume of pension paid due to revaluation and to the effect of the rate of return on the capital in the up-scenario or interest on loans in the down-scenario.

It is possible to go deeper by analyze the distributions at different time giving also some results in terms of capital requirement. In particular in the next part we focus on distribution of the random variable W_t at time $t = 30, 60, 90$. In the tables we report the results concerning the risk measures seen in section 5.2.2 applied to our capital W_t.

From the table above we can see the increasing pattern of the variability at different years. For the year 90 we have reported also the VaR and $TVar$ measures. This last in particular shows the expected amount of shortfall given the confidence of level showed in the top of the table. We had not reported this measures at lower duration since the there was not any shortfall.

Is noteworthy that the Var and $TVar$ measures are close to the annual Car and $xTVar$ respectively, this is due to the expected value of the random variable W_{90} which is really close to 0.

[34]See paragraph 6.2.1

		99.5%	99%	85%
$t = 30$	CaR	7.79	4.25	2.60
	xTVaR	8.27	5.62	4.10
$t = 60$	CaR	50.98	34.42	21.62
	xTVaR	59.56	42.74	32.57
$t = 90$	CaR	282.35	204.71	129.84
	xTVaR	340.24	250.12	190.76

Table 8: Results of risk measures at different year, results in 100'000.

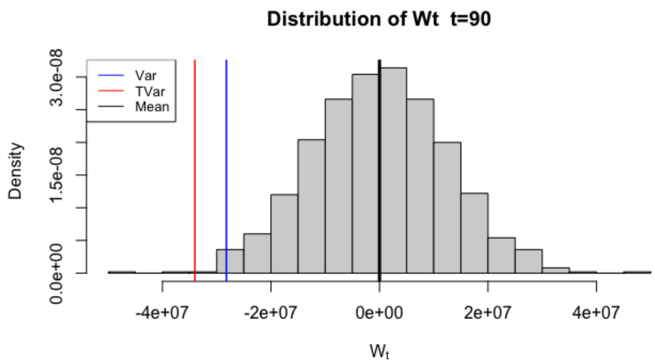

Figure 17: Empirical distribution of W_t at year 90, risk measures calibrated on a confidence level of 99.5%

In the next image we show the effect of introducing an initial capital of W_0 to our pension fund, calculated through the risk adjusted capital necessary to cover the loss highlighted in section 5.2.2 with $TvaR$ risk measure, confidence level of 99.5%.

As we can see the paths of simulation of our capital over the years has shifted up with an increase from 0 to 420000 of our *initial capital* W_0[35]. The initial

[35]The change in initial capital cannot be seen by the picture since the scale of the behavior of W_t is large.

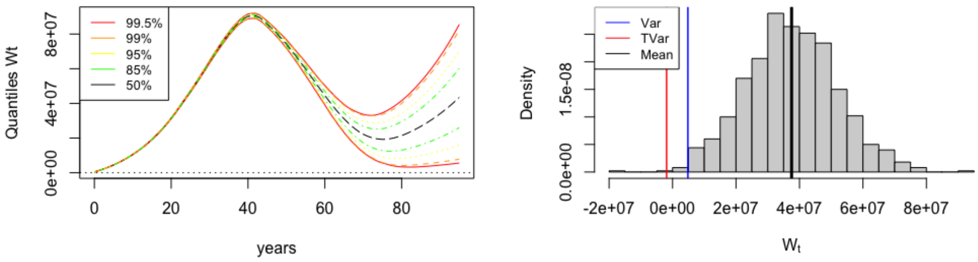

Figure 18: On the left the empirical quantiles of W_t, on the right the empirical distribution of W_{93} with risk measures calibrated on confidence level equal to 99.5%, initial capital $W_0 = 420000$

risked based capital has been calculated according to the previous step described in eq. 110. In particular the maximum shortfall correspond to the year $t = 93$, which actually coincide with the runoff of the cohort.

The introduction of a risk based initial capital has permitted to reduce drastically the *empirical ruin probability* at run-off.

In the case with $W_0 = 0$ at run-off, the paths *under* the ruin barrier were 498 on 1000 simulations, with an empirical ruin probability of $\frac{498}{1000} = 0.498$ while with $W_0 = CR_{99.5\%,T}$ we have a ruin probability of $\frac{2}{1000} = 0.002$. The results are in line with the theory behind it, in particular it is normal to expect a probability of ruin close to 50% in the absence of appropriate starting capital and without any *safety loading*. In fact, as we pointed out earlier, the expected value of capital resources at run-off, in this conditions is equal to 0.

Another factor that deserves to be brought to attention is the skewness of the distribution of our capital over the years.

In particular, we must remember that our random variables are the result of algebraic sums of many independent multinoulli processes. We know that for high t cash-outflows are closely linked to the survival/death of pension recep-

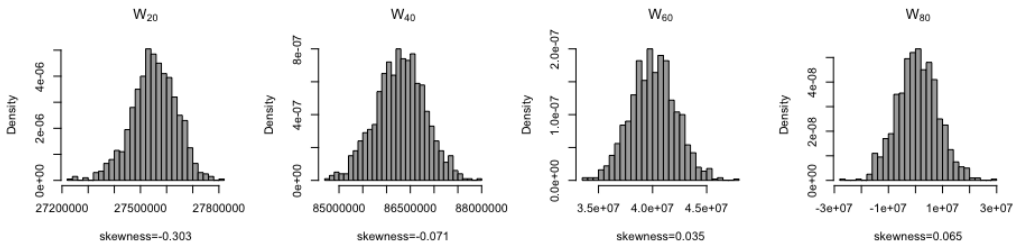

Figure 19: Distribution of W_t at increasing years t; the skewness at early years is negative, then moves to positive values as the years pass.

tors, direct or indirect. In this regard, for large values of t we can approximate our set of random variables W_t to a linear transformation of a binomial process. Once all the member are retired, the overall process can be seen as a sum of independent Bernoulli distribution.[36]

We highlight this concept since the skewness of the W_t distribution varies monotonically over the years from negative to positive. This is closely related to the trend of mortality rates with $x \to \omega$ and with the precise definition of skewness in the Bernoulli distribution:

$$\frac{1-2q}{\sqrt{q(1-q)}} \tag{124}$$

In fact we can observe that the skewness of a Bernoulli distribution is positive for $q < 0.5$ and negative for $q > 0.5$. In particular, for t greater than 43 all contracts have reached retirement status and therefore the variable W_t is solely linked to the trend of the *negative variable* cash outflow CO_t (see eq. 97).

The result of these conditions is that as the years of projection increase we

[36]A contract linked to a pensioner of state 3, with annual probability of survive of p and probability to leave a family θ, at the end of the period, the contract will be in force with a probability $p+\theta(1-p)$ or will be closed with probability $1-(p+\theta(1-p)) = (1-\theta)(1-p)$ with $p+\theta(1-p)+(1-\theta)(1-p) = 1$. Obviously the same holds for state 2 contracts. The activity of contracts belonging to indirect pension recipients, is instead directly linked with the probability of survival of recipients.

see a skewness of W_t that goes from negative to positive over the years. This fact is significant in the analysis because, in addition to the variation of the expected value and the variance, we have a further factor that affects the ruin probability, the skewness of the distribution. In this regard, a positive skewness such as that which occurs at particularly high years leads to a greater concentration of probability on negative scenarios.

Salary function: Let us now compare the result deriving from a pension scheme with the same characteristics and the same composition of policyholders, but with a different salary based function. In the previous case we considered a salary function equal to the average of the salaries paid during the entire contribution period. In this case we see the difference in terms of benefits assuming a salary function which considers only the last salary earned by the member:

$$f(w) = w_T \tag{125}$$

where T denote the total seniority of a worker despite which is the decrement considered. In this regard, w_T symbolizes the last salary earned by any member.

Despite the salaries collected remain the same, with the change of the salary function we have obviously a strong increases in the amount of benefit paid. In particular the change is made at time of computation so we have a strict difference in the computation of the first pension. This is due since, according to our assumption, the salary increases constantly over time, the average function so weight the final salary with the previous one which will be likely lower. We can relate to the fact that an average salary function is more coherent in

		S	B	B2	B3	B4	B5
Average	EPV	$342,427$	$17,913$	176	$16,420$	389	928
Salary	Premiums		0.052	0.001	0.048	0.001	0.003
Last	EPV	$342,427$	$31,822$	278	$29,288$	615	$1,640$
Salary	Premiums		0.093	0.001	0.086	0.002	0.005

Table 9: Comparison between two salary function in a defined benefit scheme, in the first part we see results and premiums amount in a *average salary function*, in the second part we have the results concerning a salary function of the type *last salary*.

a fixed contribution rate framework such the one in force in the Italian social security, since it consider the change in time of the salary. Though in a private

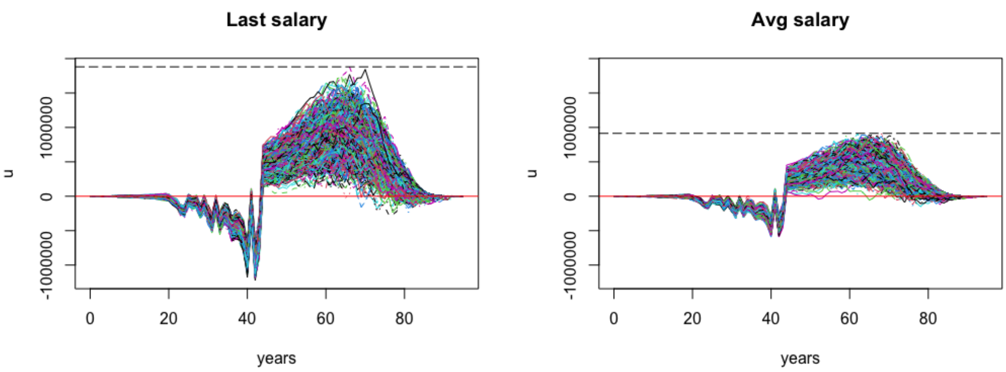

Figure 20: 1000 Simulations of the industrial profit projection seen in eq. 101, on the left-hand side we can see the incidence of benefits, premiums and construction of the reserve under the last salary function for computation of benefits; on the right-hand side we can see the lower magnitude in case of average salary function.

framework such this one, a more generous salary function obviously is strictly related to an higher premium rate. According to this, as we can see from the table 9 the overall premium is almost double, since also the total benefit are definitely greater. We can notice that the difference its much more important in old age retirement pension rather than the disability pensions. In this regard we have a smaller difference in $B2$ compared to $B3$ since the incidence of the transition from the average salary function to the last salary function is lower because the seniority passed in state 1 is also lower.

Technical basis: Up to now we have assumed a premium calculated on the medium scenario, the same one used to simulate the performance of the fund's capital.

In a more realistic analysis, an additional component also called *safety loading* is added to pure premiums. In the actuarial field relating to the length of human life, safety loading is typically implicitly included in the pure premium through the use of more stringent technical bases. In this regard, from the results analyzed previously, one possibility could be to use of low mortality scenario to price the annual contribution (I^{st} technical basis) and then use the medium mortality scenario, less severe in terms of disbursement, as a pseudo-realistic scenario (II^{nd} order basis). In this regard, we remind you that the low mortality scenario is characterized by a greater longevity, therefore by a greater total amount of the expected value of benefits paid, and therefore by a greater premium paid by the members of the fund. In further analysis we will assume the medium scenario as II^{nd} order basis and the low as the I^{st} order basis; furthermore we denote with λ the implicit safety loading applied to price the contributions paid by members, such as:

$$\lambda = \frac{p_x^{TBI} - p_x^{TBII}}{p_x^{TBII}} \tag{126}$$

so with a $\lambda > 0$ an higher contribution will be requested to members with respect to the realistic assumption of the fund, while with $\lambda < 0$ a lower contribution is requested and so a loss is expected. If the I^{st} order basis are equal to the II^{nd} order basis then, $\lambda = 0$ and the expected runoff profit is also 0.

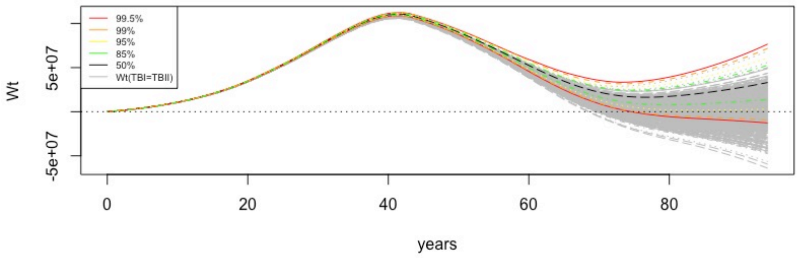

Figure 21: Simulations paths relative to the capital W_t without a safety loading (grey) and quantiles relative to W_t with TBI more severe than TBII.

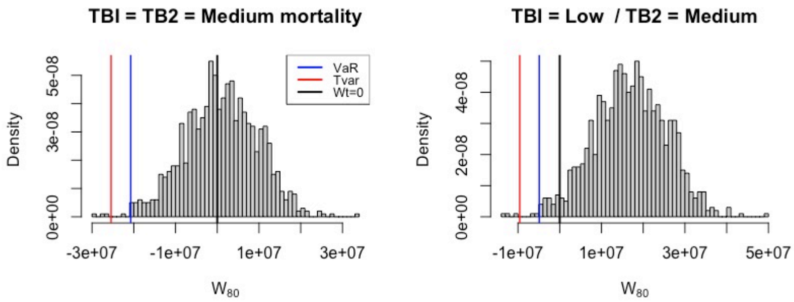

Figure 22: Distribution of W_{80} with and without safety loading. The colored lines represents the Var and $TVar$ risk measures, the black line represent the $W_{80} = 0$ scenarios.

By introducing the new TBI, the premium rate thus goes from a value of 0.0545 to a value of 0.0553 (net percentage increase of 0.08%). Although this value may seems negligible in terms of contributions paid by members, the effect on the probability of ruin observed in our analyzes is very large. In this regard,

there is a positive shift in our capital W_t, in particular at remote projection years. This is obviously due directly to the increase in the total amount of contributions requested from payers which thus allows for a greater provision of resources at the time of payment of contributions. From the figures 21 and 22 is possible to see the difference between the previous scenario. Especially from the histograms relative to the empirical distribution of W_{80} we can clearly the improvement in term of ruin probability and so on the necessary capital requirement to cover the risk of default at $t = 80$.
In this regard, we summarize the characteristics to be highlighted in the next table:

	λ	$Mean$	Sd	$TVaR_{99.5}$	p_{80}^R	$RBC_{TVaR_{99.5}}$
W_{80}	> 0	16813.3	8751.0	-9581.15	3.2%	193.3
	$= 0$	980.9	8759.4	-25'442.42	46%	513.3

Table 10: Results inherent to the empirical distribution of W_{80}, results in thousands of euro.

The probability of ruin, shows a significant reduction moving from 46% to less than 4%. This, as already mentioned, is given by a positive shift of the distribution which in fact does not impact the variability as can be seen from the standard deviation column. The $TVar$ risk measure naturally follows what has been said so far, however it is possible to see the savings in terms of the capital required to cover the risk.It is in fact more than halved.
Up to here we have considered for the definition of a safety loading only the use of different tables for the definition of the contribution quota. However, as we have seen in section 4.3, there are several assumptions, impossible to determine ex-ante, which actually affect what the premium rate. Among these, the main components that have a strong effect on the determination of the premium are the inflation and rate of return. It should in fact be noted that

by defining the rate j^* used to discount the expected value of the benefits, one implicitly recognizes a guaranteed rate of return to the members. In this regard, through a high discount rate it is assumed that the assets we collect and set aside in the mathematical reserve will accrue such yield j^* on average annually. Given that in a defined benefit scheme, of salary-based type, there is no direct link between the value of assets set aside and the actual benefits that will be paid to pensioners, an effective return lower than that recognized to members would be dangerous for solvency of the fund. In this regard, it is customary in insurance practice to insert a gap between the recognized rate and the expected one, so as to protect against any unfavorable scenario. Another strong assumption concern the revaluation of pensions. In particular, in a framework such the ours, where we intend to link revaluation to the annual inflation rate, assumptions related to the future development of inflation become crucial.

As we have described, the inflation rate in our model affects two factors: wages and benefits. As for the amount of wages, we have that, with lower annual inflation, we also have a slowdown in wage growth. Having defined a salary-based approach, we have that the amount paid as the first pension is directly related to the salary function, be it positively correlated to the salaries of each member. To this end, changes in growth of salaries deriving by slowdown of inflation are not decisive in our case, as lower wages and therefore lower contributions also correspond to more modest pensions. Instead for the revaluation of pensions, this factor is of crucial importance. In fact, an inflation-linked revaluation also requires assumptions relating to future development of inflation. An underestimation of it, with the same accumulated contributions, would lead to a strong imbalance of the fund. In fact, assuming a deterministic rate of return on assets, there would be a situation where the amount of contributions collected by the members, plus the financial returns, would not be adequate to cover the expenses deriving from the payment of pensions. In this regard, it is therefore possible to introduce further elements of prudence, thus assuming a rate of return and revaluation of the first order different from those of the second order, in order to reduce the likelihood (and magnitude) of shortfalls and what we have defined as probability of ruin. In the figure 23

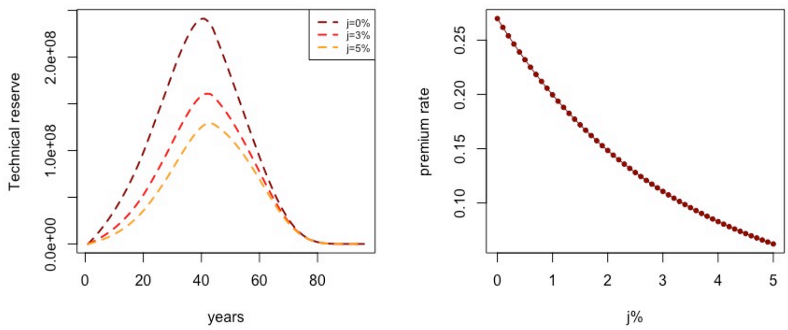

Figure 23: Reserve and premium rate as the value of the first-order technical basis relative to the financial return rate changes.

it is possible to see the effect that a change in the first order rate of return has on the technical reserves and the premium rate respectively. As regards the reserve, the greater the difference between the first-order rate and the second-order rate, the higher the value of the technical reserve set aside to meet future commitments. In fact, assuming a lower j_{TBI}, we assume a lower discount rate, affecting both contributions and future cash outflows. In this regard, given the different timing between contributions and benefits, with a reduction in the discount rate there is a greater increase in the net present value of pensions rather than that of contributions. In this regard, a greater amount of contributions is requested to be set aside to cover future out-flows.

Defined Contribution For what concern the results of a *Defined-contributions* scheme, we report below the *Transformation coefficients* computed according the same basic assumption used for the defined benefit fund.

	c_2M	c_4M	c_3M	c_2F	c_4F	c_3F
26	0.056	0.183	0	0.061	0.305	0
27	0.057	0.184	0	0.062	0.314	0
28	0.058	0.185	0	0.063	0.309	0
29	0.059	0.131	0	0.064	0.213	0
30	0.060	0.131	0	0.065	0.209	0
...	...	...	...	...	...	...
50	0.082	0.130	0	0.087	0.179	0
51	0.083	0.126	0	0.088	0.174	0
52	0.084	0.123	0	0.089	0.171	0
53	0.085	0.125	0	0.090	0.173	0
54	0.085	0.129	0	0.091	0.180	0
...	...	...	...	...	...	...
64	0.097	0.129	0	0.104	0.190	0
65	0.099	0.134	0	0.105	0.198	0
66	0.101	0.135	0	0.107	0.201	0
67	0.104	0.137	0	0.108	0.207	0
68	0	0	0.060	0	0	0.055

Table 11: Transformation coefficients computed on the assumptions made for the base scenario described above; divided for type of benefits, age of pensioning, and sex of the registered member.

From the table of transformation coefficients we can see how the trend, mainly for state 2, is consistent over time, in fact we have that as we age we have a greater portion recognized to members of the paid-up capital, this happens since obviously expectation decreases. As regards the coefficients of group 4, the trend is not monotonic as a factor that strongly affects the amount of pension paid is the probability of leaving the family. As we have seen from the formulas in section 4.3.2, this information was used here to price the amount recognized as the first pension. Low probability of leaving the family will in fact occur at very low ages, this directly affects the annuities and therefore affects inversely the transformation coefficients. A further difference can be seen in the transformation coefficients differentiated by sex. In this regard, what we might expect (and is actually found in the coefficients of group 3), is that a higher longevity, typically observed in female subjects, corresponds to a lower coefficient. This is correct and in fact longevity directly affects the quotas, however we must remember that a different trend is found in the mortality rates of disabled people and the probability of maintaining retirement rights for a widower/widow is even different. Regarding the latter, we observe (fig. 24) that the compound probability of surviving and not getting married again is much higher in the case of women, in fact the data shows a greater propensity in men for a new marriage, in particular at advanced age. In projecting the performance of the defined contribution fund, we assumed an annual contribution equal to that obtained through defined benefits, in this regard we obtained, as expected, the same results. In fact, the assumptions used are the same and consistent over time and therefore in a very basic scenario like this the result of the projection of funds with initial capital equal to 0 is identical. To compare, we report below the stochastic trend of the total amount of the fund at defined contributions in 1000 simulations.

Note that the total accumulated capital M_t is the sum of the stakes of each individual member. At the end of the accumulation phase, for each member,

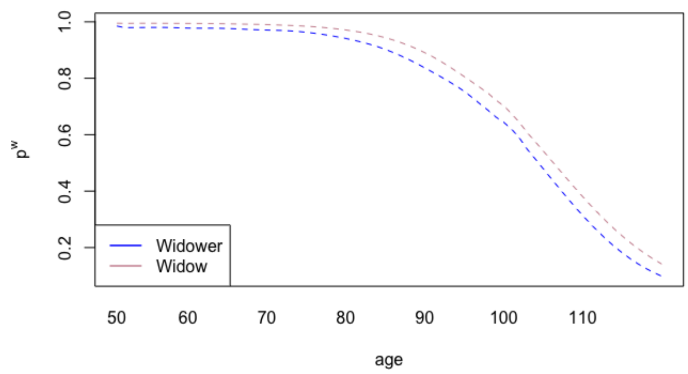

Figure 24: Annual probability to hold pension rights for a survivors, divided for widows (survivors of male workers) and widower (survivors of female workers).

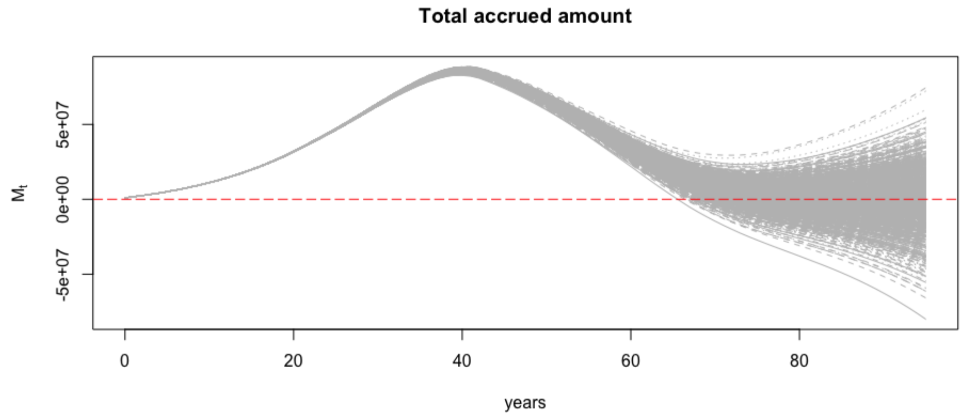

Figure 25: Accrued capital projection, 1000 simulations over the whole cohort run off.

the first pension is calculated using the product between the accumulated amount and the transformation coefficient relating to the retirement age and the type of pension to be paid. Again, in relation to the individual, the amount after the start of the provision of pensions, is revalued annually at the rate of return and reduced each year of the pension sum paid. The result is that in the absence of prudence, the accumulated capital is expected to be such that it precisely covers the amount of benefits. However, it is evident that this process is susceptible to variability. In fact, at the runoff, on the basis of the mortality actually observed, there may be a capital surplus or a shortfall.

6.2.2 Stochastic inflation

In this section, we have analyzed the results of our defined benefit pension scheme in a situation of stochastic inflation. To this end, we compared the base scenario ($\lambda = 0$) with a new set of simulations where, the annual inflation used to calculate the increase in wages and the revaluation of pensions is obtained by simulating the Ornstein-Uhlenbeck stochastic process seen in section 6.1.2. We specify that in this case the revaluation of both wages and pensions is obtained on the moving average of annual inflation. The moving average has been introduced in order to have a more realistic application, in particular in macroeconomics is well known the rigidity of salaries to inflation rate, while as regards the revaluation of pensions, an annual application of inflation would be too variable and far from the assumptions, therefore it is often preferable to use the moving average on past observations of some inflationary index.

In the simulation of the inflation process, the parameters used are:

- $\theta = 2\%$
- $k = 0.5$

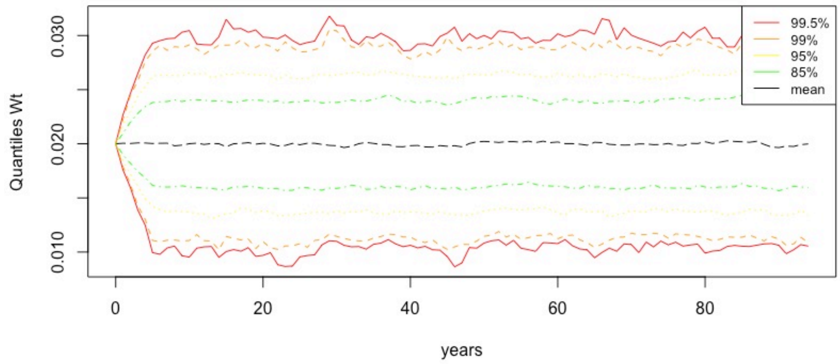

Figure 26: Quantiles and mean related to the simulation of annnual inflation used to revaluate pension and salaries in this application.

- $i_0 = 2\%$
- $\sigma = 0.5\%$

the long run average θ has been set equal to the initial inflation in order to not change the trend assumed also in the basic scenario. In particular, in this exercise we wanted to focus on the volatility that a stochastic inflation adds to the process. Expecially, the introduction of a long-term average different from the initial value would have led to a positive or negative process drift depending on the sign of $(\theta - i_0)$. Putting these two parameters equal, the annual average of inflation between the different scenarios remains so close to our basic assumptions, fixed at 2%.

Obviously, under different assumptions, for example an increase in annual inflation, an adequate adjustment of the parameters is possible. From the figure 26 it can be seen that the annual volatility of the process grows very quickly and then settles at the equilibrium value $\bar{\sigma}$ defined in the eq. 114; the mean though remain as expected stable over the years. From the figure 27 can clearly seen the effect of adding to the process a stochastic inflation rate. This

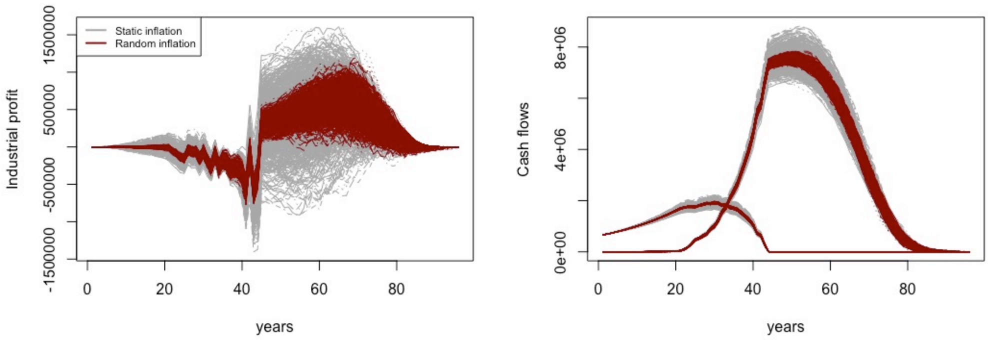

Figure 27: On the left the comparison between industrial profit simulations according to static inflation (in red) and random inflation (in grey), on the right the comparison between the cash flows in both cases.

obviously has a negative effect on possible shortfalls, in fact the probability of being in particularly serious scenarios has increased. However, thanks to the assumptions made on the inflation process, this situation is also reflected in the up scenarios. So the overall average remains unchanged. This can be seen in the following table 12 and in figure 28, where the various moments of the distribution of capital are compared.

From the table we want to highlight the strong difference in the CV of the distribution, in fact, while the mean is almost unchanged, the standard deviation increases drastically. The annual probability of default is almost the same since at time $t = 80$ both distribution of the capital are very close to be centered in 0. From the risk measures we can however see how much more severe the shorfalls are in case of stochastic inflation. This is due to the presence of unfavorable scenarios both from an economic and demographic point of view. In this regard, a W_t tail scenario is probably caused by a high inflation that massively revalues the pensions paid to members, together with a demographic scenario in which a very low mortality has occurred.

W_{80}	Mean	Std. Dev.	CV	p^R_{80}	CaR		$xTVaR$		$RBC_{Car,(0,80)}$	
Inflation					95%	99.5%	95%	99.5%	95%	99.5%
Random	1042.5	23963.3	23%	0.463	41568	68861	53734	70233	838.7	1389.4
Static	980.9	8759.4	9%	0.460	14399	21709	18514	26423	290.5	533.1

Table 12: Comparison between the characteristics of the distribution and risk measures of fund's balance at $t = 80$ with static and stochastic inflation, initial capital equal to 0, values in thousands of euro.

The likelihood of this unfavorable scenarios consequently impact also on the

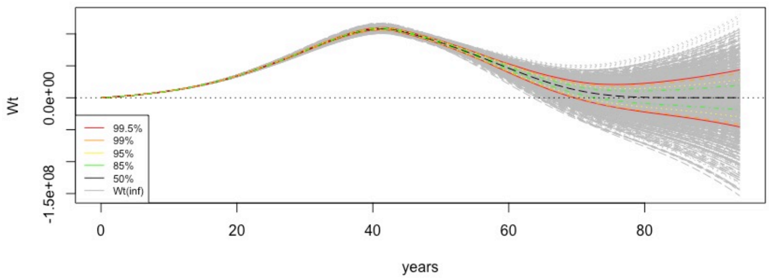

Figure 28: Simulation paths of the fund's balance over time for the process with stochastic inflation in grey, comparison with the quantiles of W_t in a static inflation enviroment (colored).

risk based capital (RBC) to be held to be solvent. Taking as reference the capital requirement calculated through a CaR approach on the time horizon $(0, 80)$, the requested capital to not be in ruin with a confidence level of 99.5% is almost three times lower in the static inflation scenario.

6.2.3 Market risk through stochastic rate of returns

Up to now, we have assumed that the rate of return on the accumulated capital by the insurance is static. This assumption is obviously far from realistic considering the high volatility of the financial markets that is observed daily. In this regard, it is possible to introduce this additional level of stochasticity by simulating the return on the accumulated assets of the pension fund for each period. This naturally brings us back to the observations made in the first part of this chapter, section 6.1.2, with the related assumptions. In particular, here we assume the independence between stock returns and bond returns. The possible investment sectors of the fund are three and are respectively: low risk, with a percentage of 75% invested in bonds and 25% in stocks, medium risk 50% and 50% and finally a high-risk sub-fund with an equities-bond division of 75%/25%. In figure 29 we have made a comparison between the three

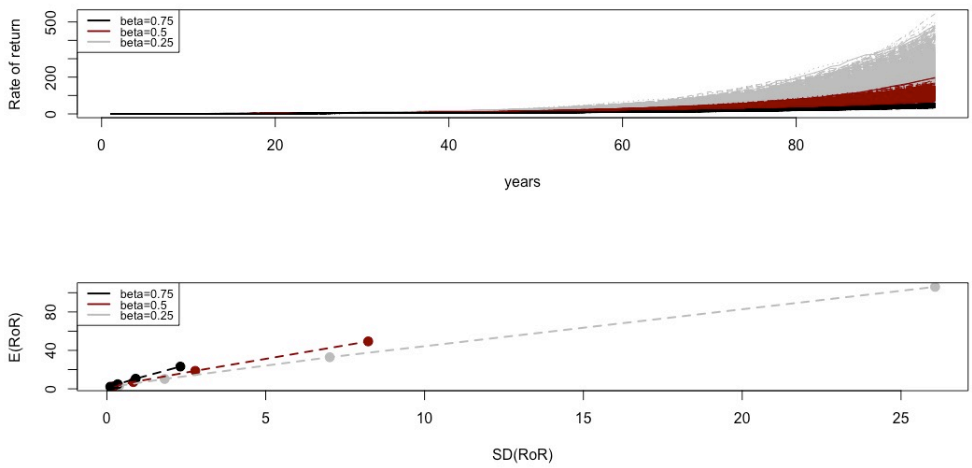

Figure 29: Comparison between three portfolios, in the first figure the simulation of multi-annual return on a initial *unit* capital, in the second figure a *risk-reward* comparison.

portfolios with different asset composition. In particular, from the first fig-

ure, we have the simulations for a unit amount capitalization according to the different portfolios. The grey one corresponds to the portfolio with a higher portion of equity while the black one to the one with a higher percentage of bonds. As we can see clearly from the maturity the results concerning the *grey portfolio* ($\beta = 0.25$) are higher in terms of profitability, though also the risk is higher since a larger variability is evident.
This relation is represented in the second figure. In particular, with the same color pattern, we represent the expected cumulative return on the unit capital on its standard deviation for times $t = 20, 40, 60, 80$. The riskier portfolio is indeed associated both with a higher return but also a very larger standard deviation of returns.
On the one hand, a higher expected return corresponds to a higher return on capital and therefore a higher discount rate for determining contributions. However, a higher variability affects the probability of ruin and therefore increases both a possible capital requirement to ensure the solvency of the fund and, at the same time, a higher security load applied to the contributions paid by members.

To this end, we present the results on the multi-state scheme we devised. The reference parameters are represented in the following table:

Multi-state		$K = 60\%$	$w_0 = 12000$	$s = 3\%$	$c = 3$	$k = 150$
Rate of return	High Risk	$\sigma = 4\%$		$\mu = 7\%$		
	Low Risk	$\sigma = 1\%$	$\alpha = 3\%$	$k = 0.5$		
Inflation		$\sigma = 0.5\%$	$\theta = 2\%$	$\gamma = 0.5$	MA = 5 years	

Table 13: Full stochastic model parameters.

As regards the technical bases used by the fund for the definition of contributions, we have:

	TBI	$TBII$
p	Med. Scenario	Med. Scenario
j	$E(j_t)$	$E(j_t)$
i	$E(i_t)$	$E(i_t)$

Table 14: Technical Basis used to price contributions and simulate future cash flows.

We decide in this first application to not apply any charges. In particular, we want to highlight, in a defined benefit regime, the effect on capital of different levels of financial volatilities. Naturally, it should be remembered that among the assumptions already highlighted we have that the asset allocation is constant over time. Therefore the percentage of investment in equity or bond does not change during the runoff of the cohort.

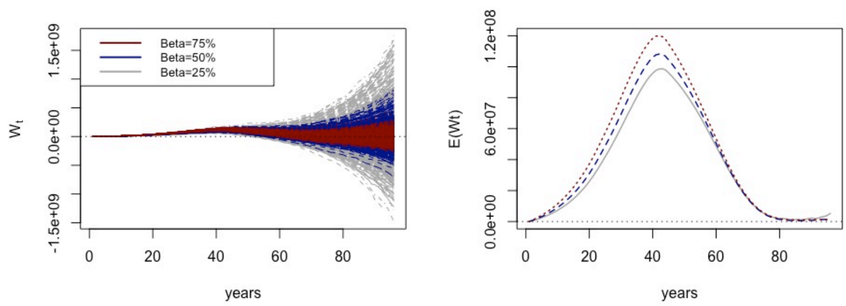

Figure 30: Comparison between capital's paths with different portfolio composition, 1000 simulations.

As we expect, if there is no safety loading and initial capital $W_0 = 0$, the expected capital at runoff converges to 0, however, as can be seen from the figure 30, to the portfolio with the largest portion of equity, is associated a higher dispersion, consequently associated with greater shortfalls in case of default. The probability of default, in this case, is not conditioned as it must be considered that the premium is calculated without any loading for prudence and using the rate of return expected from each portfolio.

Since the benefit is defined ex-ante, the contribution rate requested from the members in the three cases will differ according to the expected return of the portfolios. Consequently, the riskier, but the higher-yielding scenario will have a lower premium; the portfolio with a majority of bonds instead will consequently be associated with a higher requested premium.

Next, we compare the impact of the three different investment funds on the solvency of the defined benefit scheme in the presence of a prudent first technical basis, in fact, we introduce here a positive λ and a differential between the technical rate of return used to price the contributions and the realistic one; in particular this last is equal the second-order basis rate minus 5% while the mortality scenario used to price the contributions is the low mortality scenario.

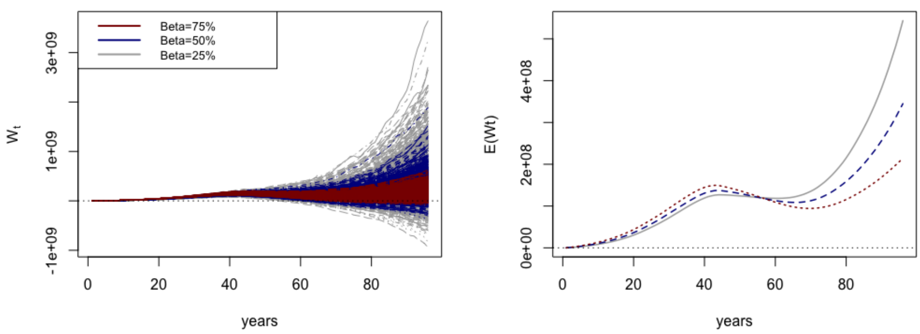

Figure 31: Comparison between capital's paths with different portfolio composition and safety loading; 1000 simulations.

The introduction of safety loading of course provides greater protection, decreasing significantly the magnitude of shortfalls. At the same time it is possible to observe how the expected capital at the runoff is clearly positive, this is given both by the surplus of collected contributions with respect to the expected benefits and by the financial return of this surplus.

The surplus between the capital value of the contributions and that of the benefits is financed by an increase in the premium, a strong variation in the premium, however, is as already highlighted, depending on the technical return rate used. To this end, for the sake of completeness, we report in the next table a comparison between the contribution rates requested from members according to the different scenarios highlighted.

β	25%	50%	75%
$E(j_t)$	6%	5%	4%
$\lambda = 0$	4.14%	5.45%	7.18%
$\lambda > 0$	4.83%	6.37%	8.42%

Table 15: Premium rates, with and without safety loading, for different asset allocations.

Let's now compare the funds with different composition of assets in terms of solvency. The following table shows the characteristics of the distribution of capital at time 95:

β	$Mean$	Sd	CV	p_{95}^R	$\text{TVaR}_{95\%}$	$\text{TVaR}_{99.5\%}$	$RBC_{\text{CaR}_{99.5\%}(0,95)}$
75%	207.10	113.60	55%	2%	-2.28	-60.78	0.94
50%	326.31	248.72	76%	7%	-103.02	-232.52	1.80
25%	508.87	518.63	102%	16%	-385.48	-735.56	$2,05$

Table 16: W_{95} distribution's characteristics, risk measures and capital requirement. Results in millions.

Note that as the portion of equity increases, the expected value of the capital W_{95} increases, but at the same time, as already seen and discussed above, there

is a net increase in variability. In particular, as can be seen from the variation coefficient, the growth of the standard deviation is more than proportional to that of the yield; in fact, it goes from a CV of about 50% to a standard deviation slightly above the average.
The latter has a strong effect on what are the probability of ruin and especially on the expected shortfall. With the portfolio 75% composed of bonds, despite the expected lower capital level, the scenarios under the ruin barrier are just 2 %, while in the case in which the portfolio is composed of equity for the most part we have a probability of default of around 16%.
The $Tvar$ applied to the three funds demonstrates how the variability has a strong impact on what is the expected shortfall. In particular, it is noted that as the confidence level increases, the incidence of risk is stronger. As regards the capital requirement, calculated according to the 110 equation, we can see how it is strictly consistent with the risk measures just seen. Note that the discount rates of the Car shown in the periods $(0, t)$ with $t = 1, 2, ...95$, is the expected return of each fund[37]. So despite the higher expected return and so the favorable discount rate, the fund with 75% high-risk investments still has a higher multi-year capital requirement. In particular, as can be seen from the figure 32, the capital requirement before a certain date is negative, as there are no paths that go below the ruin barrier. However, after a certain number of years, the cumulative amount of benefits, together with particularly negative inflationary and financial scenarios, can lead to a strong impact on capital. In this regard, it should be noted how the financial riskiness of the fund affects the timing in which the capital exceeds the barrier of 0. In particular, in the risky fund, there is an advance of almost 10 years compared to the less risky fund.

[37]The discount rate of the scheme with $\beta = 25\%$ is 6%, for the fund with $\beta = 50\%$ is 5% and for the fund with the portion of bond $\beta = 25\%$ the discount rate is 4%.

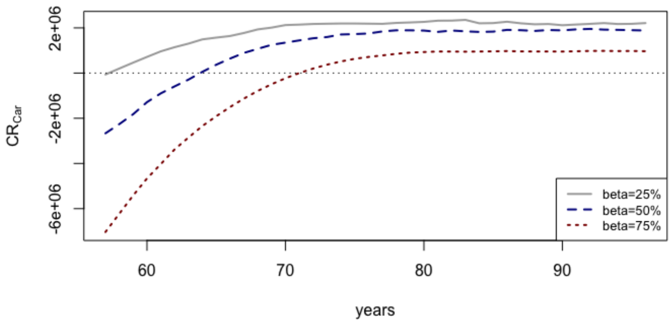

Figure 32: Capital requirements relative to the years where shortfalls occur. The values of Car are discounted to the period $(0, t)$.

It should also be noted that consistent with what has been said so far, the capital required to be solvent during the entire runoff of the cohort reflects the riskiness of the fund. Regardless of the financial component, it tends to a certain limit. Indeed, as the years pass, there is a less incidence of the demographic variable on the fund. In particular, at extreme ages, the interest rate on the debt becomes the major driver of capital. The return is therefore offset by the effect of the discount rate (assumed equal to the expected value of the debt rate) thus making the capital requirement unchanged.

Defined Contribution At this point we perform an analysis comparable to that carried out in the context of the defined benefit scheme but using a defined contribution approach. Note that the pension paid to each member is a direct function of the amount accumulated thanks to the contributions paid and the second-order rate of financial return. In the calculation of the transformation coefficients, we did not assume any safety loading related to the return rate, but rather we used the mortality rates related to the low scenario.

To this end, the parameters used are the same as those relating to the tables

13 and 14, the portion invested in bond is still defined with β.

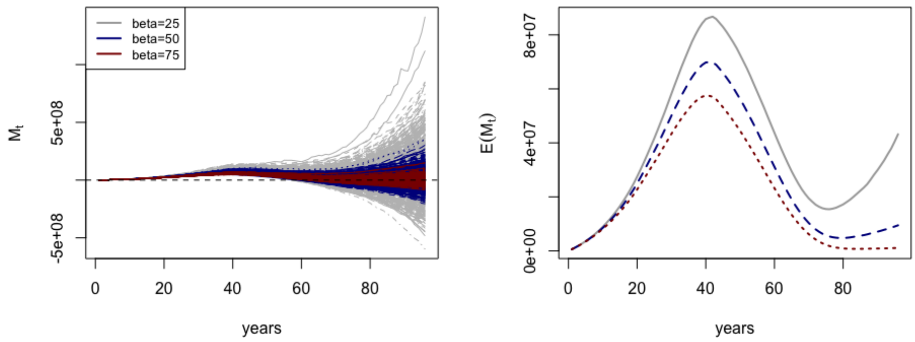

Figure 33: On the left, simulation paths related to the total accrued capital M_t, on the right-hand side the M_t mean.

From the figures 33 we can see the behaviour of 1000 simulations of M_t with different asset allocation. We considered the same cohort of entrants used in the previous cases. We remark that the expected return associated with the riskier portfolio ($\beta = 25\%$) is 6%, the balance portfolio has an expected return of 5% and the prudent portfolio an expected rate of return of 4%.

From the comparison of the two figures we can see how the trade-off between expected return and variability holds.

In the more aggressive asset allocation, a higher return on the contributions paid by the members will be expected, which will then cover a greater amount of benefits. However, even the surplus created by premature abandonment from the pension scheme accrues interest, thus transforming itself into a greater capital accumulated at the run off. On the other hand, a more careful analysis also focuses on the variability of the process. Indeed, from the figure on the left it is clearly deductible that a particularly higher expected shortfall corresponds to the riskier asset allocation. According to those two factor we cannot determine ex-ante what is the best combination from the fund point of view;

though to an higher expected shortfall correspond also an higher amount of capital to be set aside in order to be sufficiently sure to not default. As we have discussed in the section 5.2 the cost of capital must be considered in the comparison between different portfolio impact.

As in the previous paragraph, we report here the characteristics of the distribution of the total accrued amount of the cohort at year 95:

β	$Mean$	Sd	$Skewness$	CV	p_{95}^{R}	$\mathrm{TVaR}_{95\%}$	$RBC_{\mathrm{CaR}_{99.5\%}(0,95)}$
75%	1.05	29.17	0.43	27.72	51.30%	-77.79	1.73
50%	9.02	75.73	0.51	8.40	48.10%	-178.90	1.60
25%	40.54	208.93	0.86	5.170	48.53%	-450.95	$1,55$

Table 17: M_{95} distribution's characteristics, risk measures and capital requirement. Results in millions.

Recalling that in this particular case we have not added any financial prudence in the computation of the transformation coefficients, we can note how, the variability and the rate of return increase as the portion of investments in equity increases. This is in line with what we have highlighted so far. However, it must be pointed out that the relative incidence is certainly different from the scheme seen previously. In fact, according to the defined contribution scheme, in the accumulation phase, the fund does not bear any financial risk the amount of the pension is strictly linked to the financial growth of the reference capital. Therefore, any deviations from the expected rate of return are absorbed by an increase or decrease in the reference capital at the time of the member's retirement.

This leads, thanks to the higher expected value of capital given by the higher returns and an equally higher discount rate, to have an inverse situation regarding the capital to be held in order to be solvent throughout the cohort runoff.

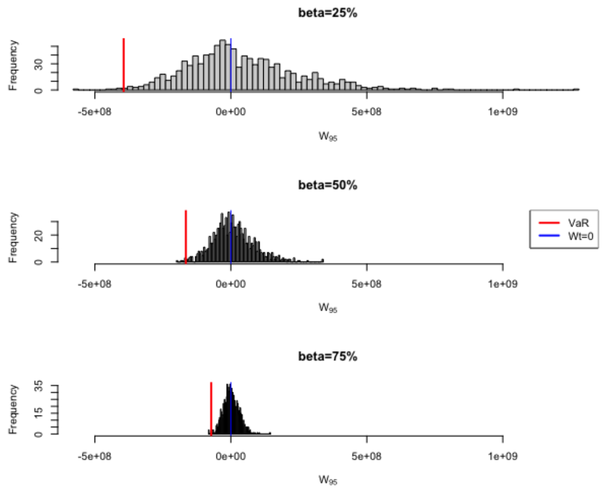

Figure 34: Distribution of W_{95} according the different asset allocations.

In fact, in the fund with riskier asset allocation, we have a higher TVaR, given by the greater variability, despite the expected value of the capital. However, as we can also see from the table 17, the capital requirement resulting from the riskiest fund is lower, albeit slightly, compared to other funds with less aggressive investments. This effect is given by the fact that the discount rate used to calculate the capital requirement is the expected rate of return.
A more current measure with what is the insurance practice would be to discount the capital to cover the risk using a certainly lower rate, the result of less risky activities. Obviously in such a situation the capital to be set aside would be strictly greater in the riskiest fund, and therefore consistent with the risk measure highlighted in the table.

6.2.4 A final comparison between alternative scenarios

In this last section, we compare the results obtained in the different scenarios and in the different types of pension schemes we set up: defined benefit and defined contribution. In this regard, with *Scenario 1* we mean 1000 simulations

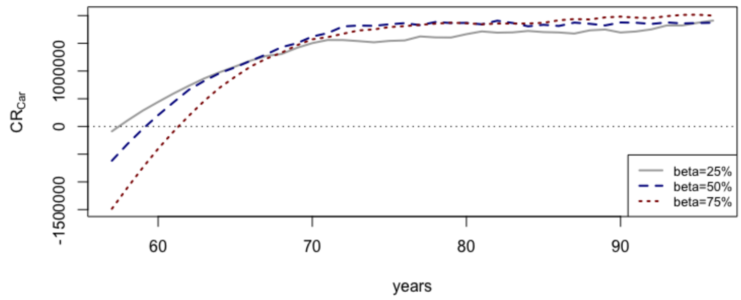

Figure 35: Capital requirements of the Defined Contribution schemes, relative to the years where shortfalls occurs. The values of Car are discounted to the period $(0, t)$.

relating to the pension fund with a single random variable that represents the permanence in the various states; with *Scenario 2* we mean the simulations relating to the fund with stochastic states and inflation; finally, with *Scenario 3* we mean the simulations related to the full stochastic model. The characteristics of the fund and the parameters used in the DB and DC funds are those described in the tables 13 and 5, while the portion of assets invested in bonds β in the *Scenario 3* is set at 50%.

For both schemes we have decided to use the following technical bases:

	TBI	$TBII$
p	Med. Scenario	Med. Scenario
j	4.9%	5%
i	2%	2%

Table 18: Technical Basis used to price contributions and simulate future cash flows in both DC and DB schemes in the next applications.

In this regard, following the procedure seen so far, we report the characteristics of the Defined Benefit fund in the following table:

DB	$Mean$	Sd	$Skewness$	CV	p_{95}^{R}	$\text{TVaR}_{99.5\%}$	$RBC_{\text{CaR}_{99.5\%}}$
Scenario 1	59.64	17.96	0.10	0.30	0%	7.25	−0.33
Scenario 2	59.37	49.36	−0.31	0.83	12%	−84.80	0.79
Scenario 3	56.52	200.11	0.36	3.54	40%	−495.05	1.76

Table 19: Scenario's characteristics comparison of W_{95} in a Defined Benefit pension scheme, results in millions.

As expected, the features shown change depending on the scenario. In fact, as we have already outlined, the variability of the process and the skewness are influenced by the underlying random variables. These moments of course directly influence the probability of default and the related risk measures. More than anything else it should be noted that the stochasticity relative to the rate of return is actually the most influential, in fact, as can also be seen from the figure 36, in the passage from the second to the third scenario there is a more than proportional increase than from the passage from Scenario 1 to Scenario 2. From the same image, we can also see the behavior of skewness over time. It is linked to the considerations already made in section 6.1.2, in fact, it tends to 0 the more the age x of the members tends to ω. However, in Scenarios 2 and 3 we add the random variables relating to inflation and the rate of return. In Scenario 2 it is possible to see how the skewness tends strongly towards 0 in the first years, this is given by the introduction of the Ornstein-Uhlenbeck process, which is a Gaussian process that strongly moves the skewness of the overall process towards 0. The same happens in Scenario 3 with the introduction of the Black-Scholes process, also distinguished by the Gaussian stochastic component, making this trend even faster than in Scenario 2.

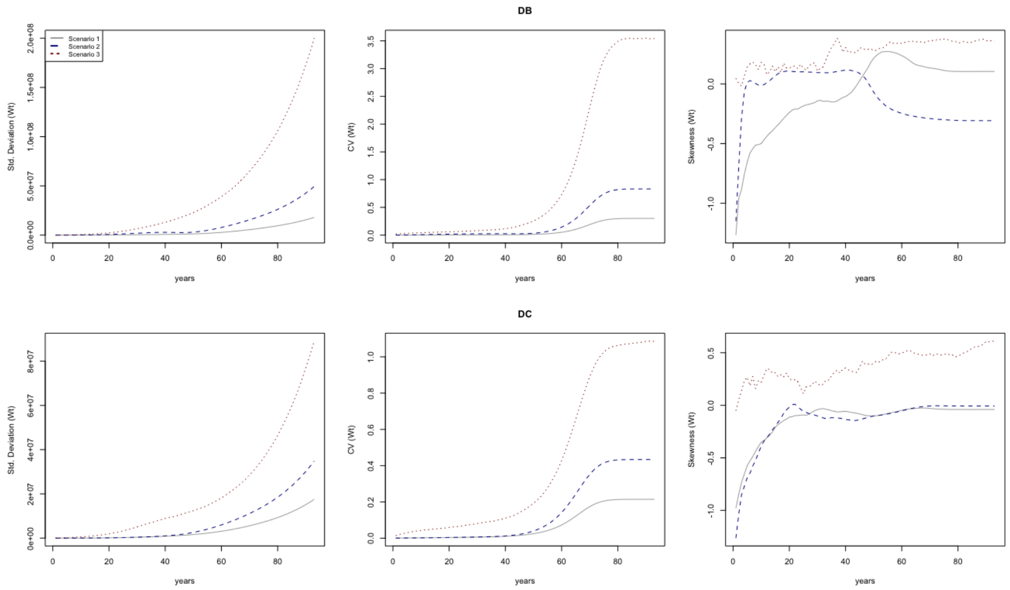

Figure 36: Comparison in time of the W_t's distribution characteristics, Defined Benefit in the first line, Defined Contribution scheme in the second line.

Speaking again of the skewness, it is possible to notice a particular disturbing effect in the skewness trend around $t = 43$. It is caused by the transition to retirement, in fact, according to the assumptions we have set, in that year we have that the whole cohort is retiring. So, all the subjects who were previously in State 1 move to State 3, passing from active contributors to pension recipients. Given the random variable W_t in eq.97, the components CI_t and CO_t are directly influenced by the random variable that describes the transition of the subjects between the States. With the transition of a subject from State 1 to State 3, (deterministic upon reaching retirement age), there is also a shift of the component relating to the Cash-flows that the *active* subjects influence, from CI to CO. Having opposite sign, the skewness of W_t is susceptible to this change, while in the standard deviation, as can be seen, there is no difference as it is insensitive to changes in sign.[38] Finally, we point out that the expected

[38] All the moments are also slightly influenced by the fact that the permanence in State 1

value, apart from some slight variations given by the simulation procedure, remains consistent for all the scenarios shown. This is naturally due to the fact that in the various scenarios the expected value of the financial and monetary components was set equal to the static value set in the previous Scenarios.

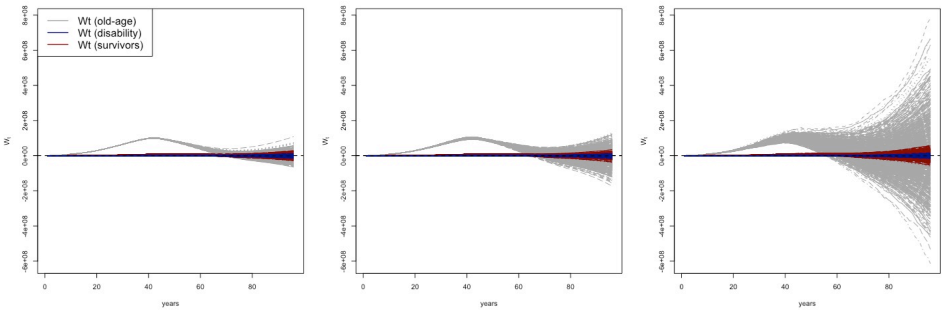

Figure 37: W_t^i behavior in different scenarios of Defined Benefit pension scheme. On the left, we have the *Scenario 1*, in the center, we have the simulation paths relative to the *Scenario 2* and on the right the *Scenario 3*.

We explore now what is the incidence of individual decreases on total risk-based capital.

We have already seen in section 6.1.2 how most of the fund's cash-outs are attributable to old-age pensions. In particular, we have seen in fact how the same premium can be divided into different components and how that relating to State 3 pensions was in this regard 91% of the total contribution rate. In order to investigate the variation of the riskiness of each individual risk in the different Scenarios, we have divided the relationship relating to the *Balance of the fund* according to equation 99, in order to study the behavior of each single component of capital.

is simulated according to probabilities $p_t^{(1,1)}$, while the permanence in State 3 is computed according to the sum of $p_t^{(3,3)} = p_t^{(1,1)}$ and $p_t^{(2,2)}$.

		Mean	*Sd*	*Skewness*	$\text{VaR}_{99.5\%}$	RBC_{W_t}	$\frac{\text{RBC}_{W_{i,t}}}{\text{RBC}_{W_t}}$
Scenario 1	W_{95}^2	0.00	4.43	-0.22	-11.41	0.27	30%
	W_{95}^3	0.00	23.76	0.15	-56.44	0.55	59%
	W_{95}^s	0.00	10.03	0.01	-25.18	0.10	11%
Scenario 2	W_{95}^2	0.00	4.42	-0.24	-11.34	0.27	16%
	W_{95}^3	0.02	48.92	-0.27	-134.58	1.31	78%
	W_{95}^s	0.00	10.27	-0.15	-26.36	0.10	6%
Scenario 3	W_{95}^2	0.01	4.69	-0.09	-12.41	0.30	6%
	W_{95}^3	3.46	178.16	0.36	-422.16	4.10	90%
	W_{95}^s	0.34	17.26	0.17	-41.97	0.17	4%

Table 20: Comparison of the characteristics of the capital, differentiated by decrements and divided by different simulation scenarios: W_t^3 is the balance relative to only old-age pensions, W_t^2 is the balance relative to disability benefits and W_t^s is relative to benefits addressed to survivors of both workers and pensioners; results in millions.

In this application, we have decided to remove all forms of prudence, in order to highlight the impact of each individual risk on the capital requirement.
As expected, regardless of the scenario analyzed, we see that the most accidental risk is longevity. In fact, the large amount of old-age retirement pensions has a strong impact on risk-based capital. Comparing the results relating to the different scenarios, it can be seen how the components react differently to the introduction of inflation and market risks. In fact, in addition to the volume, the timing of these benefits must also be considered. The disability risk, obviously linked to the amount of disability pensions, remains unchanged with the introduction of the stochastic variables i and j. It is presumably due to the fact that the settlement of benefits takes place in the early years. The

impact of adverse inflation scenarios on W_t^2 is therefore offset by the effect it has on the contributions paid. A similar effect can also be observed on indirect pensions. In fact, they are distributed over the whole period considered, given that the table considers both the pensions to survivors of workers and the pensions paid to survivors of pensioners. As regards the market risk, it has a strong impact, particularly on old-age pensions. As just mentioned, these liabilities have a high volume and a high duration and are therefore very susceptible to interest rate changes. These results are summarized in the last column of the table 20 and visible in figure 37:

In *Scenario 1*, the contribution of risk-based capital relating to the different types of benefits is balanced. However, in *Scenario 2* and above all in *Scenario 3*, the risk is polarized on old-age pensions, making the other risks almost negligible.

Finally we comment the expected value. As already mentioned, safety loading has not been introduced in this application. This should lead to an expected residual capital of 0. However, it is possible to see how *Scenario 3* carries a positive, albeit slightly, expected capital. However, it should be noted that this positive value is due to the lack of convergence in the simulation. In fact, the third scenario has a strong stochasticity and this did not allow the distribution of capital to be exactly centered in 0.

We now report the table relating to the comparison of the three scenarios for a Defined Contribution type scheme. The technical bases are the same shown in the table 18, the same used in the Defined Benefit scheme seen above. In this application, the contribution rate requested from the members was also set equal to the contribution resulting from the Defined Benefit scheme, in order to make the results fully comparable.

DC	$Mean$	Sd	$Skewness$	CV	p_{95}^{R}	$\text{TVaR}_{99.5\%}$	$RBC_{\text{CaR}_{99.5\%}}$
Scenario 1	81.77	17.53	−0.04	0.21	0%	35.20	−0.90
Scenario 2	80.39	34.87	−0.01	0.43	1%	−11.64	0.04
Scenario 3	82.09	89.08	0.61	1.09	18%	−127.68	0.41

Table 21: Scenario's characteristics comparison of M_{95} in a Defined Contribution pension scheme, results in millions.

The results are in line with the Defined Benefit's ones, in fact, as in the Defined Benefit scheme, the presence of stochasticity directly influences the variability of the overall process. As already said, this leads to a greater dispersion of W_t's distribution at runoff, and therefore the stochasticity of the inflation and rate of return directly impacts the magnitude of shortfalls.

However, it should be noted that, although the two schemes react in a very similar way to the introduction of inflation risk and market risk, the Defined Contribution scheme is more resilient and less risky in all the scenarios analyzed. This is due to the fact that the fund does not bear any risk during the accumulation period, to the detriment of contributors. Indeed, in this framework, no guarantee is provided at inception. Rather, the calculation of the first pension is directly calculated on the accumulated amount. It follows that, during the accumulation period, harsh financial and monetary scenarios lead to a reduction in the benefits paid, thus reducing the cash-outflow which negatively impacts the capital in later years.

The concepts discussed so far have a direct impact on what is the residual captal at the runoff of the cohort. Consequently, the expected capital value W_{95}, observable in the first column of the table 21, is strictly greater than the residual value in the Defined Benefit scheme, shown in table 19.

Conclusion

This thesis has studied the use of multi-state models in pension funds. Precisely, we have focused on the determination of risks and the capital requirement to guarantee the fund's solvency. Despite the numerous assumptions made, it was possible to see that the model was explanatory as regards the risk borne by the fund, both in the case of Defined Benefits and Defined Contributions. The three scenarios showed different impacts of both the standard deviation and the skewness of the target variable W_t.

In *Scenario 1*, it was possible to see the risk contribution deriving by the various decrements analyzed. In particular, the strong influence of old-age pensions confirms and underlines the importance of a correct estimate in terms of longevity. The analysis carried out in the last section highlights a high incidence of the longevity risk on risk-based capital. The longevity impact is even more enhanced when compared in a full stochastic situation. In fact, the high amount of benefits and contributions related to old-age retirement are more susceptible to financial and monetary variations; thus, making the incidence of disability and mortality risks almost negligible.

In *Scenario 2* we moved from a fixed to a stochastic inflation rate. In particular, the model has shown that this variable has a strong impact on the balance of pension funds, especially of the Defined Benefit type. The inflation, also in relation to the return on investment, can be of fundamental importance in pension funds' solvency. In such a long time horizon, a growth trend may have huge impacts due to the strong revaluation of benefits, to the detriment of moderate growth in wages and therefore, in contributions paid.

In *Scenario 3* we finally introduced a stochastic return on investments. Despite the optimistic assumptions, we have seen how asset allocation becomes a fundamental driver for both, pricing and risk-based capital. Indeed, it amplifies the dispersion of W_t's distribution observed in previous scenarios. Therefore, in a situation with a high longevity and high financial returns, shortfalls on capital can be extremely serious.

Concerning the comparison between the two types of pension schemes, we obtained results in line with those expected. Although the transformation coefficients of the Defined Contribution fund were calculated with static information, this fund was more resilient to our random variables' deviations. In fact, the accumulation structure allows the absorption of unfavorable changes in the accumulation period, maintaining a susceptibility of the fund only to the payment period.

For a more look-through inspection and the use of this model for risk-driven choices, further investigations must be made.

In further studies, in this regard, it is suggested the introduction of a dynamic asset allocation, the inclusion of dependence between financial and economic scenarios, and a greater number of simulations in order to be able to estimate the effect and evolution of all moments of the distribution of capital.

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