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Pricing Longevity Linked Securities

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Abstract

Longevity risk is a major threat to the industry of life insurance. The more time passes the more it seems that the trend of longevity continues undisturbed its growth. Policyholders live longer and receive a higher income for longer, how to contain this risk so that the lenders' probability of ruin is contained? The primary purpose of this work is to present a unique model of hedging and pricing the longevity-linked securities in accordance with the Solvency II framework in a continuous-time setting. In doing so, is required a careful study of mortality over time: looking at reality is always the best way to build a model. The future mortality will be modelled through affine processes, which will be functional and simplified in order to determine the price of the logevity-linked securities. In this context, a second generation securitisation approach is studied through derivatives contracts, analysing in the italian context the pricing of these derivatives through the Cost of Capital approach, analysing the consistency of this method with the classical pricing methods: the Wang transform, Sharpe Ratio and the Risk Neutral.

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*A mamma che mi ha sempre ricordato che chi ha le ali deve volare,
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1 Can Mortality be controlled?

1.1 Mortality history

Mortality has changed through the years, mostly in the last few decades. From historical evidence it is possible to observe an increase of life duration, which has tripled over the course of human history. Most of this increase has occurred in the past 150 years. There are not evidence that these longevity trends are slowing down. In the first half of the 18th century, the population of Western Europe was approximately 100 million, before the industrial and scientific revolutions produced many of their inventions in the field of medicine. Between the 18th century and the common era, the massive increase in population was the result of successive waves of crisis and expansion, it was not a gradual process. We are able to make a general observation of this: the huge fluctuations caused by epidemics, wars, and other disasters mark the population growth over the centuries. Observing the average annual growth rate between 1200 and 1700, it is only 1.3 per thousand. Assuming a death rate of 35 per 1,000, the birth rate will be 4% higher (see Livi-Bacci [1]). Over the centuries, the growth rate has varied greatly. In the five centuries before the seventeenth century, even if growth in two of those centuries was negative, the population of Europe almost doubled. In the eighteenth century, and especially in the nineteenth century, restrictions on population growth were lifted. The former has an annual growth rate of more than 4 per thousand, the latter has an annual growth rate of more than 7 per thousand, and the population has tripled (see Livi-Bacci [1]). The nineteenth period is rational considered an exception to the demographic transition: reductions in fertility and mortality, and changes in the life cycle of individuals and families. First, in the early industrialized countries, life expectancy began to increase, while in other parts of the world it was almost absent. Demographic analysis shows that at the beginning of the 19th century, no country in the world had a life span of more than 40 years. Over the next 150 years, certain parts of the world have made great progress in health. In 1950 the life expentancy for newborns was already 60 years old in Europe, North America and Japan (see Roser et al. [2]). The decrease of infant mortality was crucial for the enlarge of longevity, but the life expentancy, as we will see soon, increased at all ages. These improvments were obviusly signs of progress; after millenia in terrible condition, was the first time in human history that is achieved sustained gains in health for entire populations.

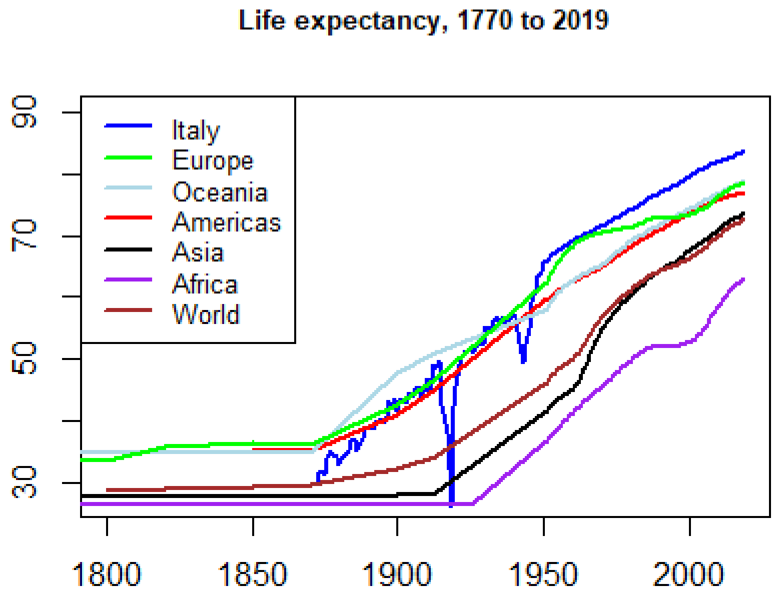


Figure 1: *Life Expectancy, 1770 to 2019, Source: Riley(2005), Clio Infra(2015), UN Population Division (2019)*

• Note: Shown is period life expectancy at birth, the average number of years a newborn would live if the pattern of mortality in the given year were to stay the same throughout its life.

This graph illustrates that the health change started at various times around the globe; worlwide the longevity rose from an average of 29 to 73 years in 2019. If we study the survival rate, shown for women in the Europe map below we can easily see that the frequency that an infant reaches at least the age of 65 is over 90%

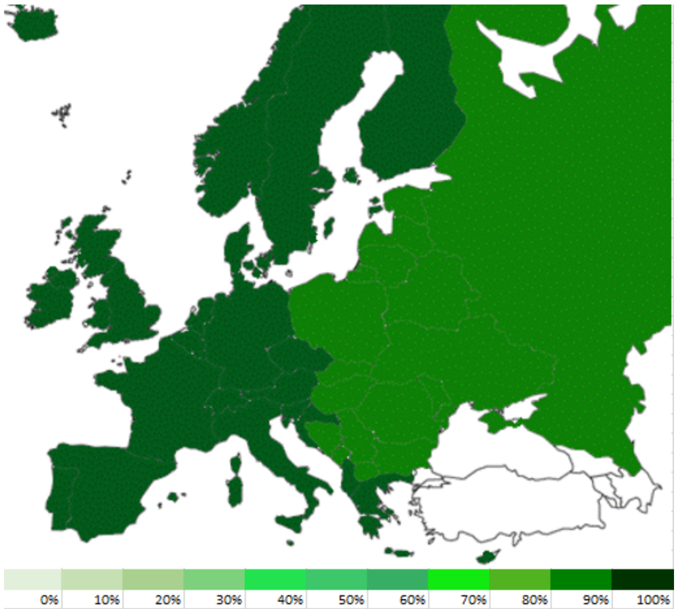


Figure 2: *Share of women expected to survive to the age of 65, Europe, 2016. Source: World Bank*

• This measure shows the share newborns that would survive to the age of 65 if subject to the current age specific mortality rates.

Often, is contended that the longevity across the world has rised only because the infants mortality fall, but this is untrue, life duration has grown at all ages. The following visualization shows the estimates projections of the remaining expected life years for a 10-years-olds; the rise, shows that the increasing of life expentancy isn't only related to the declining of infant mortality, but the mortality rate at higher ages declines as well.

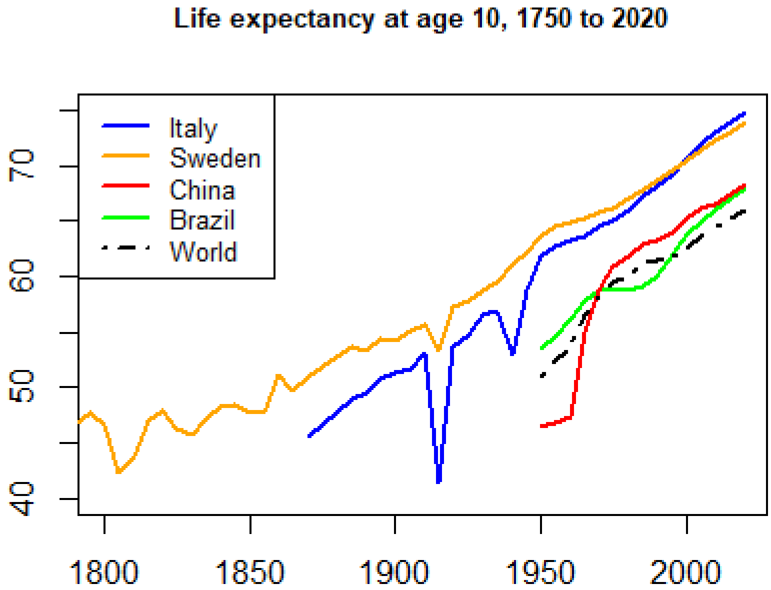


Figure 3: *Life expectancy at age 10, 1750 to 2020. Source: United Nations Population Division and Human Mortality Database 2015*
• Shown is the number of remaining years a 10 year old is expected to live. From 2015 onwards the UN mid-variant projections are shown.

We can considering now the fact that, if it is true that the life expentancy is increasing globally, is also true that there is a bound with the years lived with disability. Healthy life expentancy has grown all over the world, due to better healthcare and therapies, and this has increased the number of years, on average, that people live with disability or chronic disease. This escalate has been slower than the increase of healthy life expentancy (Roser et al.[2]). The map below depicts the number of years a person can live a healthy life and the number of years a person can live with a disability, in Italy.

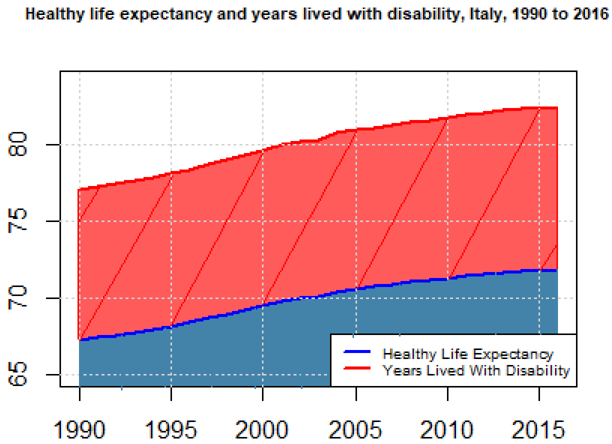


Figure 4: *Average life expectancy of an individual born in a given year, disaggregated into the expected number of healthy years, and the number lived with disability or disease burden., Italy, 1990 to 2016. Source:IHME, Global Burden of Disease*

Talking about the longevity in Italy, we can observe that it increased about 20 years between the 1950 to nowadays; looking at Istat's data 2019 (something could change due Coronavirus disease in the next report) the life expentancy is respectively, at birth, 83.2 years (female 85.4, male 81.1), at 65 years old, 21 years (female 22.6, male 19.4), at 85 years old, 6.7 years (7.2 female, male 6). Meanwhile the average age at death is 81.4 (female 83.9, male 78.5) and the Lexis point 89 (female 90, male 87). Italy is one of the most long-lived population across the world. However, the increase in life expectancy, coupled with the decrease in the birth rate, which has occurred over the last century and has led to a marked change in the structure of the population. This will consist of an increasingly large proportion of elderly people: the forecasts made by Eurostat until 2060 indicate that, in the European Union's populations, to a significant decrease of the so-called "youth population"(people in an age range 0:14) and of the so-called "active population" (people in an age range 15:64), there'll be a large increase in the number of citizens over 65. This leads to an increase over the years in the demographic dependency ratio(65^+), which is an index given by the ratio between the "non-active" population and those of the "active population". As a result, fewer and fewer workers will have to keep an increasing number of people of pensionable age, arriving in 2060 in a situation where there will be almost one elderly person to support for each worker. According to calculations made by Eurostat, in 2060 in Italy, this ratio will be 56.6%. In particular, according to Istat's forecasts, in Italy there will be a numerical overtaking of the share of the population over 65 compared to the "active" population in 2030 approximately and this gap will become greater and greater in the following years. Therefore, the fact that a person who reaches retirement age at 65 years, still has an average life expectancy of 21 years ahead has negative consequences, as more funding will be needed for both health care and pensions. This could have a negative impact on the budget of pension funds, leading to financial unsustainability. And it is precisely here that the longevity risk plays a huge relevance, that is, the risk that these institutions will have to provide annuities to an increasing number of people and for a period of time that might be higher than expected, because members of the collective could live longer than originally estimate. In order to be able to distribute the annuities granted in the future, the payers will have to set aside, from year to year, the provisons on the premiums paid by the members, which, however, might prove inadequate in relation to the annuities to be paid, in case of estimation errors on the average future life.

1.2 Insurance and mortality

The previous section highlighted how the evolution of mortality in Italy and in many developed countries has led to the formation of a phenomenon never studied or encountered previously. This occurance takes the name of longevity risk, that is the uknwon process originating from the uncertainty of the evolution of mortality in old age. Specifically, longevity risk can be defined at two different levels: individual and aggregate (see Stallard, [3]). On an individual level, longevity risk refers to the eventuality that an insured person survives longer than planned by the insurance company. At the aggregate level, on the other hand, the longevity risk occurs when, in a portfolio of insurance policies, is found an higher average survival rate than the assumed one. In this sense, it could be said that there is the presence of longevity risk when the theoretical expectations referring to mortality deviate from the empirical evidence. ANIA (National Association of Insurance Companies) defines it as "the risk of loss or unfavorable variation in the value of insurance liabilities, deriving from changes in the level, trend or volatility of mortality rates, where a drop in the mortality rate gives rise to an increase in the value of insurance liabilities ". The peculiarity of life insurance companies derives from the inseparable link of business activities with the duration of human life. This is evident in the second part of Article 1882 C.C., concerns life insurance and expresses the fundamental role of the "event relating to human life". The presence of this strong link with human life can also be extended to pension funds, where the trend in mortality in old age takes on even more importance. The insurer's benefits therefore depend largely (or totally) on the insured's random life span. In this context, the probabilistic evaluations of these random phenomena constitute a fundamental tool for the purpose of a healthy and prudent use over time of the resources collected by the company. In order to preserve sufficient profitability and capital strength, insurance companies are forced to carefully select models to represent the demographic dynamics of their policyholders, since the estimation of future exposures depends on them. In addition to this purely corporate objective, European insurance companies must comply with an equally stringent constraint in order to adequately describe the phenomenon of mortality. In fact, with reference to the principles issued by the International Accounting Standards Board (IASB), all sources of risk to which a portfolio is exposed must be used to calculate the fair value of the liabilities to be recognized in the financial statements. In addition, as part of the prudential supervision system, the new Solvency II solvency regime is based on an integrated risk approach (Integrated Risk Analysis), which requires an assessment of liabilities at market values (Market Consistent). In a insurance portoflio can be identified two sources of risk:

- investment risk
- demographic risk

The financial risk is related to fluctuations in the return rates that occur on the market and can cause a depreciation of the value of the investments made by the insurance company. As is well known, its nature is a systematic risk component. Demographic risk, on the other hand, is further divided into two components: insurance risk and longevity risk. The first is a consequence of accidental deviations of the number of deaths from the expected value and its effects can be mitigated by the increase in the number of policies in the portfolio (pooling risk). The second occurs when there are improvements in the mortality trend, which can cause a systematic deviation of the number of deaths from the expected value.

Taking now into consideration the demographic dynamics, the mortality trend observed in the last decades highlights three significant aspects:

- rectangularization
- expansion
- higher levels and strong dispersion of accidental deaths at young ages.

The rectangularization (particularly marked after 35 years of age) consists in a greater concentration of the probability distribution around the Lexis point, which in turn tends to coincide with the maximum life span ω. The expansion consists instead in the random forward movement of the Lexis point, which determines the uncertainty of the amplitude of the rectangularization (following figure). These marked trends translate into an extension of life expectancy, except for a strong volatility in mortality rates in young age due to accidental causes.

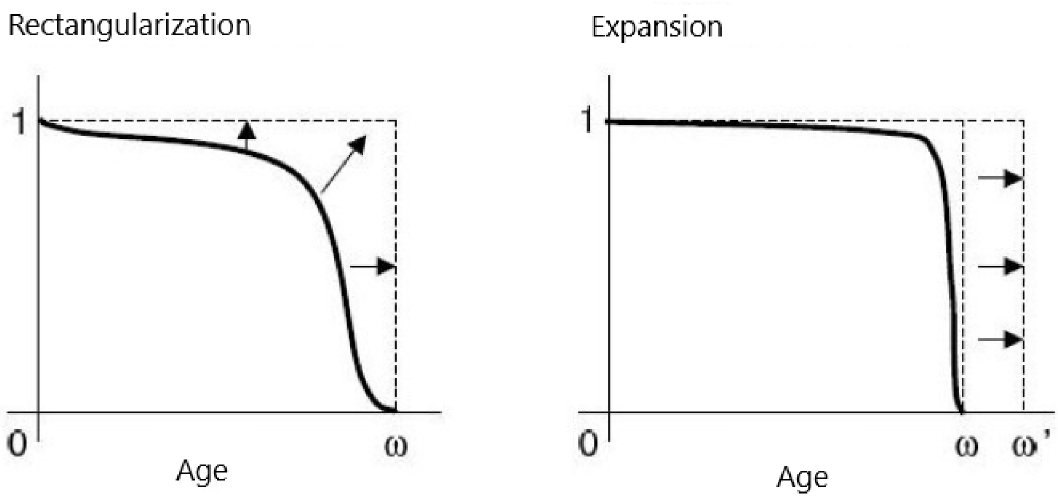


Figure 5: *Survival function trend*

The problem of possible systematic deviations between the frequency of death observed ex-post and the assumptions on the survival of the policy-holders formulated ex-ante by the insurance companies (longevity risk), can seriously compromise a prudent company management. Thus emerges the problem of addressing and managing this risk so that it does not have a strong negative impact on the company's balance and its prospective sustainability. The main control tools currently available are projected mortality tables and Longevity Linked Securities. With regard to the latter, it will be illustrated in greater detail later in the discussion. The mortality trend can instead be "anticipated" by the insurer through the use of projected tables. In a deterministic approach, making assessments with a projected table can lead to a reduction in risk compared to assessments made in the hypothesis of static mortality. But this analysis does not take into account that the projection itself is affected by randomness. It is therefore appropriate to evaluate a portfolio in the hypothesis of uncertainty of the projection (stochastic approach), (see section 2): this is a model risk, called projection risk. Insurance companies are obliged to accurately quantify the impact of randomness in the mortality trend on the different coverage offered, in order to manage the resulting risk. In fact, longevity risk affects life insurance with opposite trends compared to death insurance. With regard to policies that pay benefits in the event of life, the improvements in survival cause the increase in payments that the company has to settle and, consequently, the reserve is undersized at every moment of time. If, on the other hand, we consider the insurance companies that pay sums to the beneficiaries in the event of the death of the insured, the extension of life implies a containment of costs in favor of the company. Furthermore, the overestimation of costs negatively affects the business management activity. Due to this overestimation, the company sets aside availability to cover future costs that it will not incur, by immobilizing assets that could be used differently.

1.3 Longevity Risk: introduction and implications

For an annuity-paying institution managing the future mortality and so its modelling is a priority. As stated before, mortality rates are affected by two types of risk: idiosyncrastic risks, namely unexpected changes within a homogeneous risk class, and systematic changes involving the whole community (see Bauer et al.[4]) the latter are the most dangerous for annuity-paying institutions: in fact, in this case if the community lives more than assumed in the assessments, mortality affects in the same direction for all members making the longevity risk difficult to mitigate. On the contrary, the accidental variations are mitigatible by increasing the collective: the variations around the average are absorbed for the Law of Large Numbers. Doing the same for systematic mortality leads instead to a worsening condition. The longevity risk is linked to the receipt of an annuity of an insurer who lives more than calculated on the technical basis. Thinking about public welfare, the transformation coefficient is linked to future mortality, but it still remain exposed to longevity risk; is impossible to have reliability for a long time. The longevity risk is therefore the risk of paying an annuity for longer than estimated, the issue is who will pay the rent for the extra years of survival. In the public pension scheme reinsurance the State itself with more taxes; but for private pension-paying institutions the situation is different. In England between the years 1990 and 2000 they risked default. In Italy at the beginning there was less attention because the market was not yet very developed, now that companies begin to pay the first annuities are aware of it. Longevity risk is hard to measure because, although using projection models, the evolution of mortality remains uncertain. Uncertainty can be reduced, but it cannot be undone: however sophisticated the model may be, it will never be able to capture future mortality for certain. The exposure is to three different types of risk:

- process risk, although it can be predicted the model well and estimate the parameters well, then the individual does not always behave on average, it will be observed a fluctuation around the expected value
- parameter risk, parameters are estimated on the observed reality, but this is only one of the possible realizations of the random variable mortality
- model risk, the model does not represent well the evolution of mortality, there will always be deviation between observed and predicted mortality.

This risk, as stated in the previous subsection, involves annuity-paying istitutions like insurance companies, social security institutions, which provide basic social security benefits in capital or annuity (including reversibility), closed and open pension funds providing benefits in capital or annuity. With particular reference to :

Life Business

- In life case insurance, a higher survival rate than expected involves payments by the insurance company to the beneficiaries or insured for a larger number

of periods and, therefore, an increase in actuarial liabilities; in fact, the company will be forced, to against this risk, to set aside higher reserves on the premiums paid by the policy holder;
- In death case insurance, since the company will be required to pay a certain amount to the beneficiaries in the event of the death of the insured person, a longer life span of the member is on the one hand positive for the company as it will obtain a reduction in the benefits it will have to provide, on the other hand, an overestimation of costs may be a negative factor in the management of the holding, since monetary amounts that could be used in other assets are set aside.

Defined benefit pension plan

Longevity risk is already present in the accumulation phase because an unexpected extending of life expectations will lead to a greater loading because the annuity is pre-established and may not be reduced to compensate a longer duration. Moreover, the risk of longevity is not easily defined ax-ante, because if the contributions paid and the returns realised are not sufficient to ensure the predetermined annuity, the fund may require payments additional overtime, which may weigh in whole or in part on the adherent or may be financed by the fund. Is also hard to determine what is attributable to the longevity risk, the investment risk or other risks.

Defined contribution plans

In the accumulation phase, the risk of longevity is usually borne by the individual adherent, while it does not have an economic impact on the manager. Infact the conversion coefficient of the amount in rent, usually, is not established at the time of registration of the insured/member, but at the beginning of the annuity. Any improvements in life expectancy will result in a coefficient of conversion into a less favourable pension and therefore into a lower pension rate. In doing so, however, there is a reputational risk for social security institutions, if they change the conversion coefficient of the amount in annuity when the member retires.

The Equitable Life Assurance Society (ELAS), the world's oldest life office, was forced to close to new business in December 2000, bringing the potential implications of longevity risk to public attention. ELAS sold with-profits pension annuities between 1957 and 1988, with guaranteed annuity

rates set by reference to specific assumptions about interest rates and life expectancy. These embedded options became extremely lucrative in the 1990s as a result of a combination of declining interest rates and improved mortality, and it was the increase in the value of these options that caused ELAS's financial difficulties. These could have been prevented if ELAS had secured its vulnerability to both interest rate risk and longevity risk, but ELAS failed to recognize the magnitude of its possible exposure for several years. The inability of ELAS to do so reflects the Society's poor interest in longevity risk management.
The management of a pension scheme, in order to comply with the principle adequacy, requires a prior assessment of the impact of the longevity risk on future transformation coefficients. At the stage of retirement, there isn't usually a review of the benefits as a result of unexpected changes in life expectancy, therefore, the longevity risk is borne by those who directly provide the annuity (insurance companies, pension funds,...). To protect its own solidity, an insurance company (or a pension fund) has different ways. We will analyze these in the following subsection.

1.4 Managing Longevity Risk

The first strategy a paying institution could implement is to manage risk on its own: to do this, are used mortality tables projected in such way as to determine the actuarial values of the annuities to be paid in the future. Are used projected mortality tables (in Italy A62 built by ANIA), and there is a security loading that could be implicit in the technical basis, or explicit. The loading is surely needed when the conversion coefficient is guaranteed. However, the use of appropriate mortality tables (obtained through stochastic projections) does not eliminate the risk of longevity, although it may help in its management. To mitigate longevity risk, in an annuity portfolio, there are different techniques that we can consider:

- Natural Hedging
- Capital Requirement
- Traditional Reinsurance
- Financial Reinsurance (securitization)

1.4.1 Natural Hedging

Firstly, when a company sells both life insurance and annuities, a hedge can be created. Natural hedging is the practice of tying the prices of two lines of products to the mortality (death rates) and longevity (survival rates) levels, respectively, so that their opposing movements in payoffs will balance each other. The idea is to cover "naturally" longevity with insurance objects that are inversely related to life span, through an appropriate mix of life and death case benefits. There are some advantages. Infact firstly is not required a counterparty, then there are not transaction costs, and also it could be an internal risk diversification tool. We can combine a death event with an annuity payment. Observe that it is possibile only if the institutions offers also death case policies, otherwise for implementing a natural hedging, the institutions must give the longevity to an insurance company that offers also death events. But note that in doing the natural hedging strategy there is a great criticality: people who agree a death case policy have an age different from whom is exposed to longevity risk, there are huge difficulties to reach a satisfactory compensation; besides the death cover generally ends when the person has a retirement age (65 years old). It is needed a lifetime death case, but who subscribes this kind of policy has different features from who suffers longevity risk.

1.4.2 In the Solvency II Directive

Solvency II is the directive, which has been under discussion for over ten years, that updates the prudential regulation of European insurance companies. As the banks also companies must maintain a cushion of additional resources to guarantee their business. With Solvency II, which follows the same approach already done in the banking sector (Basel III) and in US insurance regulations, the cushions of protection asset are no longer calculated on a fixed basis, but change to relation to the actual risks of the company (risk oriented system, both technical and investment risk). The project developed is based on a multi-level scheme defined Lamfalussy. The Directive reviews the prudential supervision of the insurance sector, following a risk-based approach: the company in its businesses will have to take into account all risks to which it is exposed and the interrelationships between all of them, also taking into account the risks on the asset side (total balance sheet approach), managing these risks effectively and efficiently. Firms will eventually be able to deter-

mine their capital requirement through the use of an internal model subject to the approval of the Supervisory Authority.
In particular ex **Article 75 (Valuation of assets and liabilities)**:
Member States shall ensure that, unless otherwise stated, insurance and reinsurance undertakings value assets and liabilities as follows:

(a)assets shall be valued at the amount for which they could be exchanged between knowledgeable willing parties in an arm's length transaction;

(b)liabilities shall be valued at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arm's length transaction.

When valuing liabilities under point (b), no adjustment to take account of the own credit standing of the insurance or reinsurance undertaking shall be made.
Specifically ex **Article 76 (General provisions)**:

1.Member States shall ensure that insurance and reinsurance undertakings establish technical provisions with respect to all of their insurance and reinsurance obligations towards policy holders and beneficiaries of insurance or reinsurance contracts.

2.The value of technical provisions shall correspond to the current amount insurance and reinsurance undertakings would have to pay if they were to transfer their insurance and reinsurance obligations immediately to another insurance or reinsurance undertaking.

3.The calculation of technical provisions shall make use of and be consistent with information provided by the financial markets and generally available data on underwriting risks (market consistency).

4.Technical provisions shall be calculated in a prudent, reliable and objective manner.
A huge relevance has furthermore **Article 77 (Calculation of technical provisions)**

1. The value of technical provisions shall be equal to the sum of a best

estimate and a risk margin.

2. The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure. The calculation of the best estimate shall be based upon up-to-date and credible information and realistic assumptions and be performed using adequate, applicable and relevant actuarial and statistical methods. The cash-flow projection used in the calculation of the best estimate shall take account of all the cash in- and out-flows required to settle the insurance and reinsurance obligations over the lifetime thereof. The best estimate shall be calculated gross, without deduction of the amounts recoverable from reinsurance contracts and special purpose vehicles.

3. The risk margin shall be such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations.

4. Insurance and reinsurance undertakings shall value the best estimate and the risk margin separately. However, where future cash flows associated with insurance or reinsurance obligations can be replicated reliably using financial instruments for which a reliable market value is observable, the value of technical provisions associated with those future cash flows shall be determined on the basis of the market value of those financial instruments. In this case, separate calculations of the best estimate and the risk margin shall not be required.

5. Where insurance and reinsurance undertakings value the best estimate and the risk margin separately, the risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligations over the lifetime thereof. The rate used in the determination of the cost of providing that amount of eligible own funds (Cost-of-Capital rate) shall be the same for all insurance and reinsurance undertakings and shall be reviewed periodically. The Cost-of-Capital rate used shall be equal to the additional rate, above the relevant risk-free interest rate, that an insurance or reinsurance undertaking would incur holding an

amount of eligible own funds, equal to the Solvency Capital Requirement necessary to support insurance and reinsurance obligations over the lifetime of those obligations.
As stated in the incipit of this subsection, the regulator requires a locked capital to cover the risk of unfavourable operating results that are not already covered by technical provisions, ex **Article 101 (Calculation of the Solvency Capital Requirement)**:

1. ...

2. The Solvency Capital Requirement shall be calculated on the presumption that the undertaking will pursue its business as a going concern.

3. The Solvency Capital Requirement shall be calibrated so as to ensure that all quantifiable risks to which an insurance or reinsurance undertaking is exposed are taken into account. It shall cover existing business, as well as the new business expected to be written over the following 12 months. With respect to existing business, it shall cover only unexpected losses. It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period.

4. The Solvency Capital Requirement shall cover at least the following risks:
a) non-life underwriting risk;
b) life underwriting risk;
c) health underwriting risk;
d) market risk;
e) credit risk;
f) operational risk.
Operational risk as referred to in point (f) of the first subparagraph shall include legal risks, and exclude risks arising from strategic decisions, as well as reputation risks.

5. When calculating the Solvency Capital Requirement, insurance and reinsurance undertakings shall take account of the effect of risk-mitigation techniques, provided that credit risk and other risks arising from the use of such techniques are properly reflected in the Solvency Capital Requirement.

In this context the principal measures of risk are VaR_{ϱ} and $TVaR_{\varrho}$ given by:

$$\begin{aligned} VaR_{\varrho} &= min(x|P(X > x) = 1 - \varrho) \\ TVaR_{\varrho} &= \mathbb{E}[X|X > VaR_{\varrho}] \end{aligned} \tag{1}$$

The Solvency II Directive takes $\varrho = 99.5\%$ on a one year time horizon:

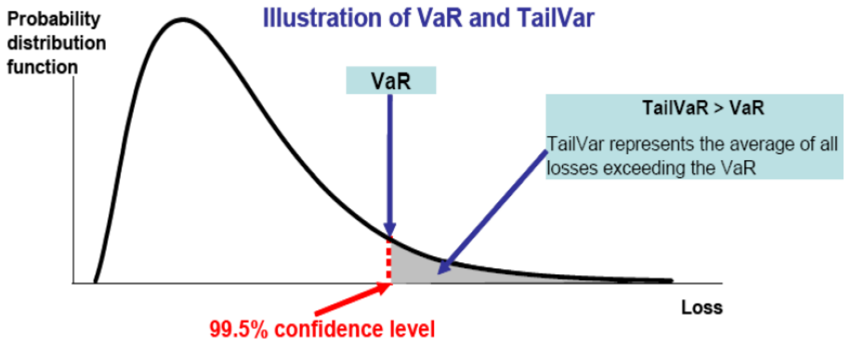


Figure 6: *VaR and TVaR*

The entire project is based on many different aspects, but as can be seen, the longevity risk is mentioned in the first pillar relating to the capital requirements of the risks considered "quantifiable". Solvency II proposes two alternative ways to calculate capital measures: Standard Formula or Internal Model. In the first case, a mathematical formula adaptable to all companies; in the second case, each company will use its own internal model approved by the supervisory authority. The objective of Solvency II is precisely to encourage companies to develop internal models, in such a way as to have a more adequate capital requirement that fully reflects the risks to which they are exposed.
Regarding the capital amount for longevity is needed the **Article 105.3 (Calculation of the Basic Solvency Capital Requirement)** that states:

The life underwriting risk module shall reflect the risk arising from life insurance obligations, in relation to the perils covered and the processes used in the conduct of business. It shall be calculated, as a combination of the capital requirements for at least the following sub-modules:
a) the risk of loss, or of adverse change in the value of insurance liabilities,

resulting from changes in the level, trend, or volatility of mortality rates, where an increase in the mortality rate leads to an increase in the value of insurance liabilities (mortality risk);
b) the risk of loss, or of adverse change in the value of insurance liabilities, resulting from changes in the level, trend, or volatility of mortality rates, where a decrease in the mortality rate leads to an increase in the value of insurance liabilities **(longevity risk)**;
c) the risk of loss, or of adverse change in the value of insurance liabilities, resulting from changes in the level, trend or volatility of disability, sickness and morbidity rates (disability – morbidity risk);
d) the risk of loss, or of adverse change in the value of insurance liabilities, resulting from changes in the level, trend, or volatility of the expenses incurred in servicing insurance or reinsurance contracts (life-expense risk);
e) the risk of loss, or of adverse change in the value of insurance liabilities, resulting from fluctuations in the level, trend, or volatility of the revision rates applied to annuities, due to changes in the legal environment or in the state of health of the person insured (revision risk);
f) the risk of loss, or of adverse change in the value of insurance liabilities, resulting from changes in the level or volatility of the rates of policy lapses, terminations, renewals and surrenders (lapse risk);
g) the risk of loss, or of adverse change in the value of insurance liabilities, resulting from the significant uncertainty of pricing and provisioning assumptions related to extreme or irregular events (life-catastrophe risk).
The Solvency Capital Requirment for longevity risk is calculated as a variation in the value of assets less liabilities due to a longevity shock. Is the difference between mathematical reserve with shock V_t^{shock} and best estimate V_t^{BE}:

$$SCR_t = \Delta BOF_t | longevityshock = V_t^{shock} - V_t^{BE} \tag{2}$$

where BOF are the basic own funds, the shock is a 20% permanent reduction in mortality rates ∀ x ages. It represent the amount of capital that a pension fund must hold for absorbing the unexpected losses related to one-year longevity risk at 99.5% confidence level. The capital that the institution sets aside, however, has to be kept locked. The fund loses by not investing it, it would be better to yield the risk into reinsurance, lowering the SCR.

1.4.3 Reinsurance

Insurance companies may decide to transfer to third parties part of the risk of their portfolio: in fact, it is possible that companies may not have the instruments to refund policyholders who have been subject to particularly unpredictable events, such as natural disasters and terrorist attacks. It is precisely here that the concept of reinsurance assumes a certain importance. It is a tool used by insurance companies to insure themselves, through the payment of a certain premium, and for this can also be called "insurance of insurance" or "second-degree insurance". In this way the insurance company leaves a substantial part of the risk it has assumed from its customers, transferring it to the reinsurance company and is, therefore, able to take policies, which would otherwise be financially unsustainable and would therefore be forced to decline. As a result, the original policyholder will be subject to a lower risk of default of the contractual counterparty and will have a higher probability of receiving exactly what is due if the event of the contract occurs. As a result, it can be said that this will encourage the future growth of insurance companies, as, thanks to this instrument, they will be able to benefit from a reduction in the capital requirements required by the supervisory authorities. So institutions that provide pensions or annuities will move (at least partially) the unacceptable longevity risk to other insurers or reinsurers. For example, insurers may pay a premium to purchase a reinsurance contract; pension funds may hedge their longevity risk by acquiring annuity products from insurance.
Reinsurance contracts that address demographic risk are:

- surplus (S)
- excess of loss (XL)
- stop loss assets (SLA)
- stop loss cash flows (SLCF)

  Surplus
  It is a proportional reinsurance cover in which the reinsurer pays a portion of the annuity each year. The fee is specific to each individual contract, and aims to make the performance of the policies homogeneous.
  - indicating with M the maximum amount that the transferor wishes to

pay for each annuity, the reinsurer will pay the amount that exceeds M;
- reduces performance volatility and random fluctuations around the expected value (pooling risk).

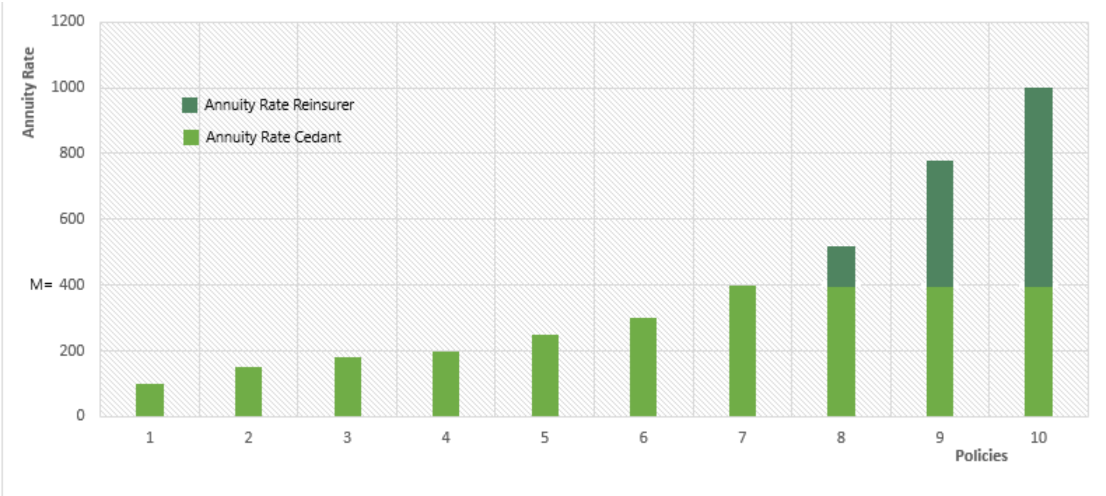


Figure 7: *Reinsurance via Surplus, M=400, from Pitacco E., Denuit M., Haberman S., Olivieri A. (2009) "Modelling Longevity Dynamics for Pensions and Annuity Business". Oxford University Press*

Excess of Loss

It is a contract in which the reinsurer pays the part of the annuity that exceeds an expiration date "m" (tail risk). The full-life annuities of the annuity provider are converted into temporary annuities, the reinsurer takes charge of a deferred annuity.
- $x_0 + m$ must be sufficiently high (e.g. the Lexis point);
- the reinsurer bears the worst part of the risk: the uncertainty about mortality is stronger in old age (longevity risk);
- strong risk margin on the reinsurance premium;

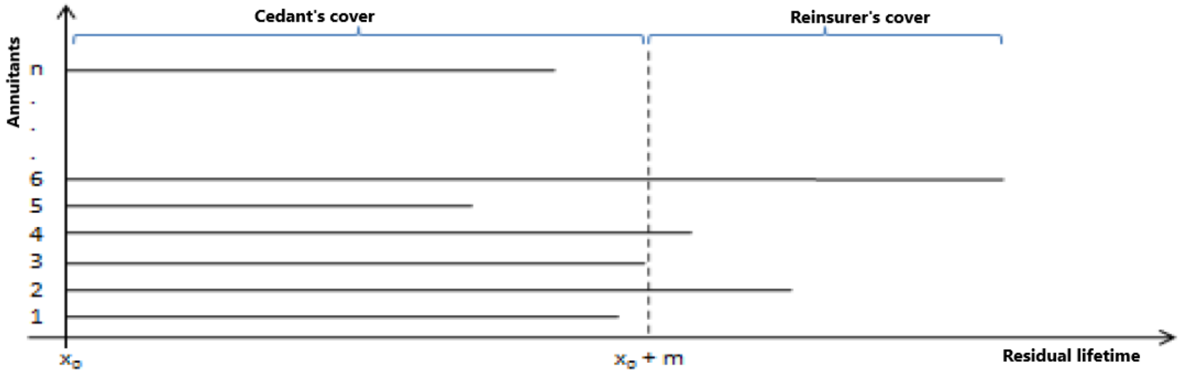


Figure 8: *Reinsurance via XL, from Pitacco E., Denuit M., Haberman S., Olivieri A. (2009) "Modelling Longevity Dynamics for Pensions and Annuity Business". Oxford University Press*

Stop Loss on Assets
It is a portfolio contract, which aims to prevent the transferor's insolvency caused by systematic deviations in mortality:
- the reinsurer's coverage is based on the comparison between the portfolio assets in a predetermined date (A_t), with the portfolio reserve required to meet the insurer's commitment at date (V_t). The reinsurer pays if $A_t < (1+r)V_t$ with 0 <r <1. Loss is defined as asset insufficiency.
- Typically defined on a reference period of short / medium duration:
- short period, emphasizes the effect of random deviations:
- medium period, generates a strong exposure to longevity risk for the reinsurer and therefore very high reinsurance premiums.

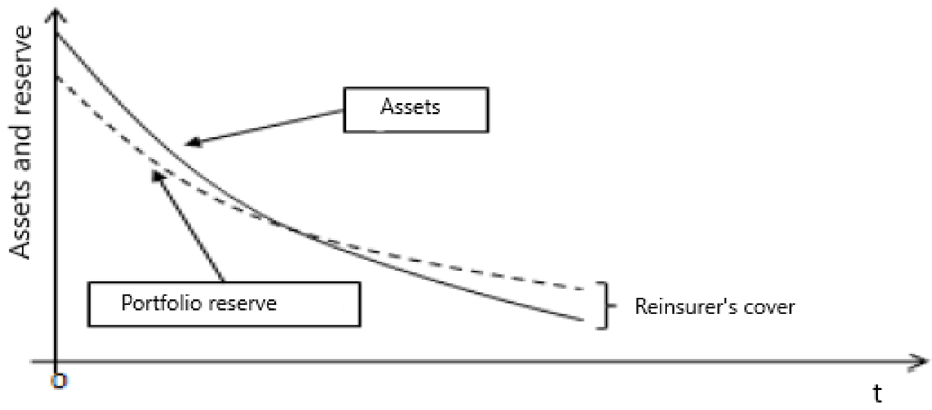


Figure 9: *Reinsurance via Stop Loss on Assets, from Pitacco E., Denuit M., Haberman S., Olivieri A. (2009) "Modelling Longevity Dynamics for Pensions and Annuity Business". Oxford University Press*

Stop Loss on Cash Flows

It is a portfolio contract, which aims to prevent the insolvency of the transferor caused by systematic deviations of mortality:

- reinsurer coverage begins when random payments of annual benefits to annuity recipients exceed a predetermined amount: $L_t^{'} = \mathbb{E}[R_t](1+r)$ where $\mathbb{E}[R_t]$ is the expected value of the portfolio payments of annuities and r>0;
- an upper limit is set for the $L_t^{''}$ reinsurance contract

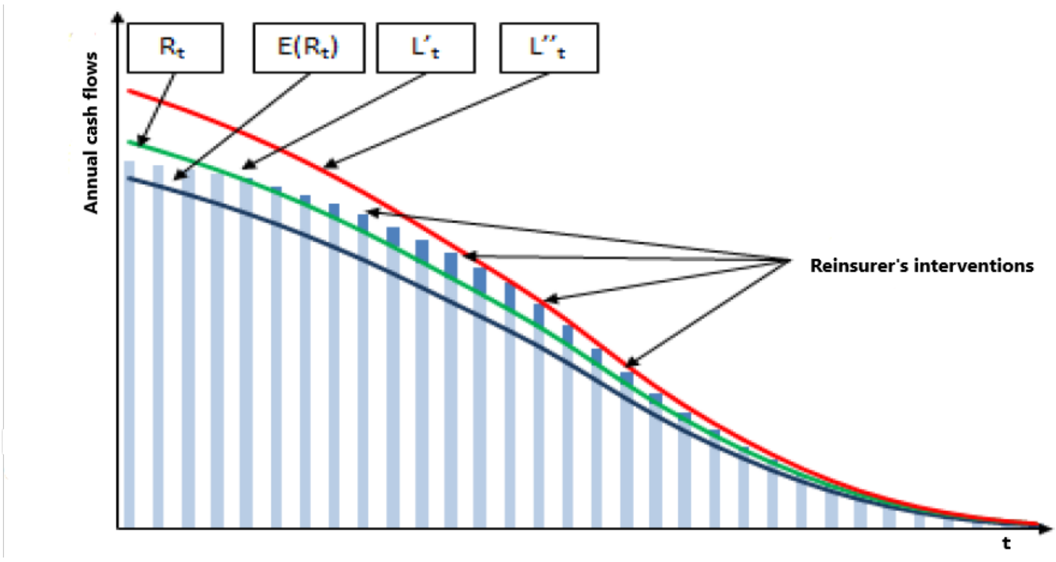


Figure 10: *Reinsurance via Stop Loss on Cash Flows, from Pitacco E., Denuit M., Haberman S., Olivieri A. (2009) "Modelling Longevity Dynamics for Pensions and Annuity Business". Oxford University Press*

Anyway note that an important condition for the protection against the risks to which the first-tier insurer is subject is that the reinsurer has a diversified portfolio, the more this will be verified, the more effective the coverage from the exposure to the risk in question. This is the problem of the implement of the traditional reinsurance for longevity risk, because also the reinsurance company cannot diversified its portfolio. A reinsurance company has in its portfolio policies that belong to insurances companies that work in different countries, but the longevity risk affects all the countries (at least those where work companies that offer annuities subject to longevity).
In the end, all of these methods have weaknesses in practice. Stricter regulatory capital requirements on mortality and longevity risk have limited reinsurers' appetite for taking the risk. Furthermore, many insurance providers lack the capital needed to offer both life insurance and annuities in the case of the natural hedging. The last method of mitigating longevity risk is to use the lifemarket, which allows longevity-linked liabilities to be exchanged.

1.4.4 Securitization

The current problems with state-run pay-as-you-go pension schemes in many countries intensify the longevity risk: many governments' implementation of tax incentives for annuitization of private wealth and reductions in future benefits from public pension schemes may lead to an increase in demand for annuities (see Kling et al.[5]). Under Solvency II, insurance companies must measure and assess longevity risks (see Levantesi and Menzietti [6]). Therefore, the level of capital required for longevity is increasing (see EIOPA [7]). Before, annuity providers could only transfer longevity to reinsurers that could provide standardized offer or customized insurance, but this risk transfer became more and more expensive, and involves a counterparty credit risk. Therefore, it is necessary to look for other alternatives to transfer longevity risks, especially securitization (see Cox et al.[8]; see Li et al.[9]). Securitization is a prominent method of managing this risk, which involves isolating the cash flows associated with longevity risk and repackaging them into cash flows that can be exchanged in capital markets (see Cowley and Cummins [10]). Several supporting instruments, such as so-called longevity or survivor bonds, have been mentioned in academic literature, Blake and Borrows in 2001 were the first to support a mortality(longevity)-linked securities. When insurance firms issue annuities, they use the premiums received to purchase matching assets, that is, assets whose cash payments fit the expected pattern of payouts on the liabilities that they face as closely as possible. In the case of level annuities, they mostly invest in fixed-income bonds. In the case of index-linked annuities, they hold index-linked bonds; no insurance firm will be willing to write index-linked annuities unless it could offset the resulting inflation risk by purchasing an index-linked bond issued by, say, the government or a utility. However, there is one risk for which there are no matching assets: longevity (or mortality) risk. A easy solution to the issue of longevity risk would be for the government to issue survivor (or life annuity) bonds, bonds whose future coupon payments are determined by the percentage of the total population of pension age (suggest, 65) on the issue date still present on the future coupon payment dates. For a bond issued in 2020, the payoff in 2040 will be equal to the proportion of 65-year-olds in the population who have lived to the age of 85. As a result, the coupon is directly proportional to the amount that an insurance provider required to pay out as an annuity to the average person with an average pension. Every year, a new tranche of bonds will be issued on a unisex basis. The Government Actuary will decide the issue price. The bonds will be available for sale on the open market and could be resold. Large occupational plans that face aggregate longevity risk will be natural buyers of such bonds (see Burrows and Blake [11]). In the same way as governments in a number of countries have assisted pension funds in insuring against inflation by issuing index bonds, the issuing of survivor bonds will assist mature pension funds in insuring against the risks associated with an aging population. The reduction in annuity expense loadings may be important. The authors do, however, point out that their idea is not new: survivor bonds are nearly identical in form to the 1759 Geneva Tontine Bond (see Cooper [12]; and Jennings and Trout [13]). However, the first attempt to issue a longevity-linked security failed in 2004. Nonetheless, the general opinion among practitioners appears to be that, though an advancement is yet to come, "betting on the time of death is set", and many investment banks, such as JPMorgan or Goldman Sachs, have established longevity risk trading desks.

Before looking deeper into these securities, it's a good idea to think about who may be involved in the markets for mortality-linked securities. There are many stakeholders in these markets (see Cairns at al.; [14])

- **Hedgers**: Hedgers are a natural class of stakeholders that have a specific exposure to longevity risk and wish to mitigate the risk. Annuity providers, for example, stand to lose if mortality improves faster than expected, while life insurers stand to benefit, and vice versa. Because of these offsetting exposures, annuity companies and life insurers will hedge each other's longevity risks. (If the annuity and life books are part of the same life office, the annuity and life books have at least a partial natural hedge.) Alternatively, entities with unfavorable exposure to longevity risk can compensate other players to reduce their risk. A life office, for example, can mitigate its longevity risk by reinsuring it or transferring it to the capital markets.

- **Government**: The government may be involved in markets for mortality-linked securities for a variety of reasons. It may wish to encourage certain markets and assist financial institutions that are exposed to longevity risk (for example, it may issue longevity bonds that can be used as instruments to hedge longevity risk). Actions of this nature have the potential to reduce the probability that major corporations would be financially ruined by their pension systems, with the result that society as a whole profits from increased economic stability. As the "insurer of last resort," the government will be left holding the bag if private-sector pension funds and insurance firms fail to meet their obligations. Furthermore the government may be willing to reduce its own longevity risk. The government is a major holder of this risk in its own right, through the pay-as-you-go state pension scheme, its obligations to provide health coverage for the elderly, and for a variety of other reasons.

- **Speculators and Arbitrageurs**: Short-term investors who exchange their views on the course of individual security price fluctuations can be drawn to a market in longevity-linked securities. The active participation of speculators is very beneficial to market liquidity and, in reality, is required for the performance of traded futures and options markets. Arbitrageurs aim to benefit from price discrepancies in similar securities. Arbitrage requires well-established pricing relationships between the relevant securities for it to be a profitable operation.

- **General Investors**: Capital market institutions, such as investment banks or hedge funds, may be interested in gaining exposure to longevity risk if projected returns are fair, since it has a low correlation with traditional financial market risk factors.

Anyway, note that there are difficulty in establishing the new market: first, the imbalance between existing exposures and the willingness of hedging providers, second, mortality-linked securities must meet the different needs of hedgers and investors (difficult to reconcile - the former require coverage effectiveness, while the latter demand liquidity), and furthermore the absence of a market price for longevity risk. Recently, some longevity derivatives have appeared. These products are based on mortality/longevity rates and are similar to products in financial markets (see Blake et al.[11]). The two main securities related to longevity proposed in the literature are longevity bonds (Blake and Burrows) and survival swaps (see Dowd et al.[15]). Longevity swaps are considered better because have lower transaction costs, and can be customized according to individual characteristic. Also, they do not need to have a liquid market. Therefore, longevity swaps appear to be the most relevant derivative to hedge longevity risks. As interest rate swaps, survival swaps can be divided into a simpler set of derivatives: S-forwards. Blake and

Burrows suggested using survivor bonds to move longevity risk to capital markets, then several writers have concentrated on longevity derivatives. The itemize sequence below lists some of the longevity-related items listed in the literature:

- **Longevity Bond** - Coupons are calculated using the survivor index (the percentage of the reference population that is still alive). (Blake and Burrows[11], Dowd[15]).

- **Longevity Swap** - Consists of swapping a series of potential cash flows corresponding to a given population's realized survival rate (based on the reference population who are still alive) in exchange for fixed survival rates agreed upon at the contract's outset. (Dowd[15], Dowd[16])

- **Mortality Option** - At maturity, the call holder receives a payout equal to $max({}_T\hat{q}_x -_T q_x; 0)$, and the pull holder receives a payout equal to $max({}_Tq_x -_T \hat{q}_x; 0)$, where ${}_T\hat{q}_x$ is the fixed mortality rate and ${}_Tq_x$ is the realized mortality rate. (Cairns et al. [17]).

- **Survivor Option** - The payoffs are similiar to the previous one, but the mortality rate is replace by the survival rate (Dowd [15])

- **q-forward** - Is a contract where is trade an amount equal to the realized mortality rate of a given population cohort (floating leg) in exchange for a fixed survival rate agreed upon at the contract's inception (fixed rate payment), at a future date T, contract maturity. (Coughlan et al. [18]).

- **S-forward** - similiar to the above q-forward, but here the realized mortality rate is replace by the survival one (Life and Longevity Markets Association[19])

These derivatives have attracted many researchers and practitioners who have published many articles in this area, but due to pricing difficulties and the fact that these products only eliminate longevity risks, they are not yet widely traded in the financial market, can be built in theory, but their implementation proves hard. In reality, there have only been a few longevity/mortality-linked securities announced or launched in the capital market. Some of these items are in the following table:

Longevity Bonds	The market has seen the issuance of two types of longevity bonds: coupon-related bonds that pay declining coupons based on a cohort survival index (EIB/BNP bonds) and principal at risk longevity bonds that pay coupons but whose principal is determined by an index calculating the difference between two mortality rates (Kortis/Swiss Re).
Mortality Bonds	Inspired by Cat Bonds, bonds issued to cover insurers and reinsurers from natural disasters. Investors lose any or all of the return and even the nominal value of the bond if the loss happens under prespecified conditions. In this way, insurers mitigate the risk of a sudden rise in mortality by sharing it with other investors. Mortality Bonds operate in the same way: the bond's principal amount is related to a mortality rate, and if the mortality rate reaches the predicted threshold, investors will lose some or all of their investment. Swiss Re issued the first mortality bonds (VITA1) in 2003, followed by other related bonds (VITA2 and VITA3) (2004 and 2007)
Longevity Options	The underlying of a call option is indexed to a longevity commodity. In 2013, Deutsch Bank introduced an out-of-the-money 10-year choice focused on a 5-year cohort of men and women aged 55 to 79 (population of England, Wales, and the Netherlands). It is an over-the-counter (OTC) commodity that allows holders to profit if the real survival rates are higher than those of the underlying asset.
Longevity Swaps	Longevity swaps can be classified into two types based on whether they have a custom-made or uniform index-based cover. JP Morgan enters into a personalized swap with Canada Life and a structured swap with Lucida in 2008.

Table 1: Longevity Products

Unlike traditional financial products, these instruments were not sufficiently appealing to investors for a variety of reasons, depending on the form of coverage. For example, the generic solutions are based on visible population indices that differs from those used by the insurer (i.e. basis risk). Evidently, unlike specialist reinsurers, financial investors are not specialists in insurance risks and, as a result, focus on publicly accessible indices to prevent manipulation. As a result, these structured products leave the insurer vulnerable to the basis risk, which can be very high. Personalised securities mitigate this issue because the longevity risk is entirely covered; however, this level of security is normally prohibitively costly. Furthermore, pricing may be the primary cause these linked securities struggled to succeed. The longevity market is still immature and incomplete, with an obvious lack of liquidity. As a consequence, there is a shortage of information for trading purposes, resulting in the absence of mark-to-market rates for longevity derivatives. Various pricing methods have been introduced in the literature, the majority of these strategies being inspired by traditional pricing methods used in the financial industry, such as the risk-neutral, Sharpe, and Wang approaches. Before investigating these pricing methods, is surely useful now to easy explain how does a longevity securitization works:

We consider for example a bond with the following characteristics:

- Issued to hedge longevity of a portfolio of immediate annuites
- Has coupons proportional to survivors of a given cohort
- Guarantees the principal repayment

The annuity provider has the requirement to pay immediate annuity to the given cohort of l_{x_0} individuals with age x_0 at time 0, name R the amount of the individual annuity, so in t the annuity privider will pay the random amount Rl_{x_0+t}; there is an exposure to the risk of systematic deviations between l_{x_0+t} and $\hat{l}_{x_0+t}$ respectively actual and expected number of survivors aged x+t at time t
The straight coupon bond gives a cash flow RC_t $\forall t$, and a repay equals to RF in T, with constant coupons, with amounts equal to RC(see Lin and Cox [20])
Through a Special Purpose Company (SPC), coupons are splitted between investors and the annuity provider in two financial instruments related on realized mortality at each future time t $\forall t, t = 1, 2, 3, ...T$. So that we will

have

$$RC_t = R(B_t + D_t) \tag{3}$$

where RB_t is the benefits recived by the annuity provider, and RD_t the specular payments to investors, i.e.:

$$B_t = \begin{cases} C & l_{x_0+t} - \hat{l}_{x_0+t} > C \\ l_{x_0+t} - \hat{l}_{x_0+t} & 0 < l_{x_0+t} - \hat{l}_{x_0+t} \leqslant C \\ 0 & l_{x_0+t} - \hat{l}_{x_0+t} \leqslant 0 \end{cases} \tag{4}$$

$$D_t = \begin{cases} 0 & l_{x_0+t} - \hat{l}_{x_0+t} > C \\ C - (l_{x_0+t} - \hat{l}_{x_0+t}) & 0 < l_{x_0+t} - \hat{l}_{x_0+t} \leqslant C \\ C & l_{x_0+t} - \hat{l}_{x_0+t} \leqslant 0 \end{cases} \tag{5}$$

So $D_t = C - B_t$

Suppose now that SPC purchases, at price W, a straight coupons bond with the characteristics described above; being P the premimium that an annuity provider pays to SPC for the hedging of his longevity risk, and V the price paid by investors to buy the longevity bond issued from SPC with coupons RD_t and face value RF. The price of the longevity bond is the expected value of the future payoffs under a risk-adjusted probability measure, named RA, is considered independence between demographic and financial risk:

$$W = RFv(0,T) + RC\sum_{t=1}^{T} v(0,t) \tag{6}$$

$$P = R\sum_{t=1}^{T} \mathbb{E}^{RA}[B_t]v(0,t) \tag{7}$$

$$V = RFv(0,t) + R\sum_{t=1}^{T} \mathbb{E}^{RA}[D_t]v(0,t) \tag{8}$$

$v(0,t)$ risk-free discount factor

The situations described is the following:

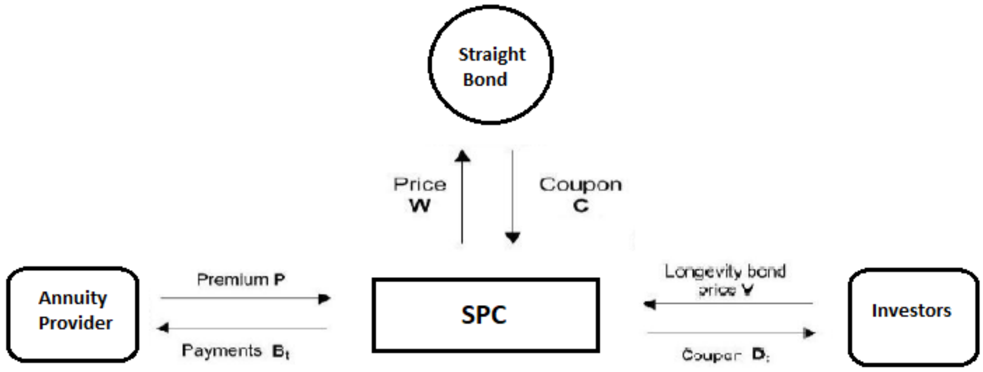


Figure 11: Longevity Bond cash flows scheme

In analogy to what stated above for the bond, is possible present the structure of a **Vanilla Survivor Swap**:

- the annuity provider must pay immediate annuities to a cohort of l_x earners aged x in 0
- fixed annuity of amount equal to 1
- $\hat{l}_{x+t}$ expected number of survivors aged x+t in t
- l_{x+t} actual number of survivors aged x+t in t
- exposure to the risk of systematic deviations between l_{x+t} and $\hat{l}_{x+t}$
- l_{x+t} - $\hat{l}_{x+t}$: losses of the annuity provider $\forall t$
- named π as the fixed premium rate of the swap, set so that the value of the swap is zero upon issue $\Rightarrow$ market value of the fixed leg = market value of the floating leg

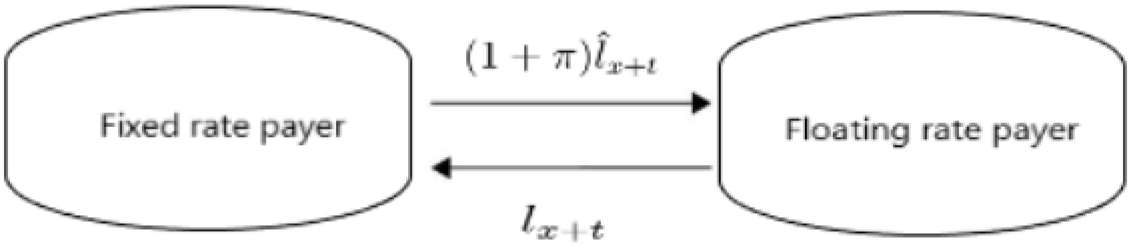


Figure 12: Cashflows survivor swap

The value of the vanilla survivor swap in t=0 for the fixed-rate payer is:

$$\mathcal{V}(0, l_{x,t}) - \mathcal{V}(0, (1+\pi)\hat{l}_{x+t}) \tag{9}$$

Remembering the hypotesis of indipendence between mortality and interes rate:

$$\begin{aligned} \mathcal{V}(0, (1+\pi)\hat{l}_{x+t}) &= (1+\pi)\sum_{t=1}^{T} \hat{l}_{x+t} v(0,t) \\ \mathcal{V}(0, l_{x+t}) &= \sum_{t=1}^{T} \mathbb{E}^{RA}[l_{x+t}] v(0,t) \end{aligned} \tag{10}$$

Where the first of (10) is the expected value of the fixed leg under the real world probability measure; the second of (10) is the expected value of the floating leg under the risk-adjusted probability measure. Is possible now to give the Premium π of the vanilla survivor Swap:

$$\pi = \frac{\sum_{t=1}^{T} \mathbb{E}^{RA}[l_{x+t}] v(0,t)}{\sum_{t=1}^{T} \hat{l}_{x+t} v(0,t)} \tag{11}$$

The cashflows of a vanilla survivor swap could be represented as:

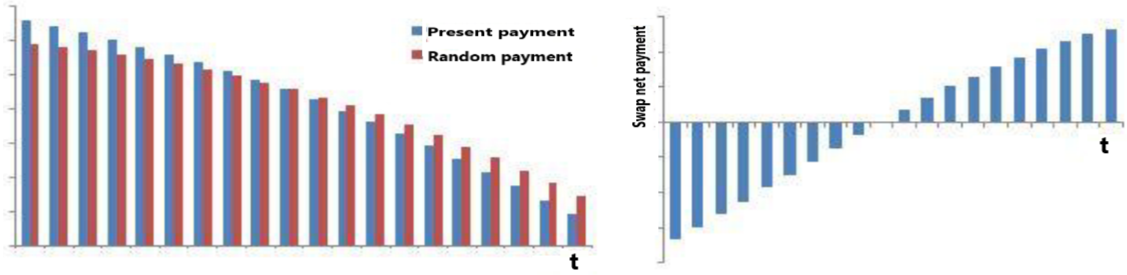


Figure 13: Cashflows survivor swap

In the following table are listed some of the issued survivor swaps:

Date	Hedger	Term (years)	Intermediary
Jan-08	Lucida	10	JPMorgan
Jul-08	Canada Life	40	JPMorgan
Feb-09	Abbey Life	Run-off	Deutsche Bank
Mar-09	Aviva	10	RBS
Jun-09	Babcock	50	Credit Suisse
Jul-09	Royal and Sun Alliance	Run-off	Goldman Sachs

Table 2: Some issued survivor swaps, from Biffis and Blake (2009) [21], "Mortality-Linked Securities and Derivatives"

The coverage for the longevity risk could be standardized or customized, in the following figure a resume:

	Advantages	Disadvantages
Standardized Coverages	are cheaper than customized; Shorter maturity (lower exposure to credit and counterparty risk)	No perfect coverages Basis risk Roll risk
Customized Coverages	Perfect coverage, no basis risk; Set-and-forget hedge, is needed a minimal monitoring	More expensive then standardized; Little liquidity; Longer maturity ; Less interesting for investements

Figure 14: Standardized and Customized Survivor Swaps

The classical pricing
There are basically three approaches

- the Wang transform (Wang 2002 [22]);
  the risk-adjusted probablities are given by a distortion operator
- the Sharpe ratio approach;
  is assumed that investors ask, for a risk premium (Sharpe ratio) equal to that required for the non-diversifiable risk of other financial instruments, to assume the longevity risk
- risk neutral approach;
  the same pricing principles used for the pricing of financial derivatives apply to longevity-linked securities

Wang defines the following distorsion operator:

$$g_\lambda(u) = \Phi[\Phi^{-1}(u) - \lambda] \tag{12}$$

with 0<u<1 and λ is a parameter that reflect the market price of risk . The real world probability distribution is skewed to produce risk-adjusted expected values which can be discounted with the risk-free structure.

$$ {}_t\hat{q}^*_{x_0} = \Phi[\Phi^{-1}({}_t\hat{q}_{x_0} - \lambda_{x_0}(t_0)] \tag{13} $$

with ${}_t\hat{q}^*_{x_0}$ is probability under the risk adjusted measure.
This approach is followed by Lin and Cox [20], Cox-Lin-Wang (2006) [23], Denuit-Devolder-Goderniaux (2007) [24], and for italian data by Levantesi-Menzietti-Torri (2010a, 2010c) [25]. Following the latter approach, it could be used the market price of the annuities for the estimation of λ:

$$ a^{market}_{x_0}(t_0) = \sum_{t\geqslant 1}(1 - \Phi[\Phi^{-1}({}_t\hat{q}_{x_0} - \lambda_{x_0}(t_0)])v(0,t) \tag{14} $$

where $a^{market}_{x_0}(t_0)$ is the annuities market price. From (14) is possible to give an explicit formulation of P and V:

$$ \begin{aligned} P &= R\sum_{t=1}^{T}\mathbb{E}^*[B_t]v(0,t) \\ V &= RFv(0,t) + R\sum_{t=1}^{T}\mathbb{E}^*[D_t]v(0,t) \end{aligned} \tag{15} $$

However, there are some critical elements as the lack of a secondary market for annuities in Italy. To solve a hypothesis is made: the market value of the annuities assumed equal to the market value calculated using the most widely used technical bases (IPS55).

The **Sharpe Ratio** is affected by the same problem. It must be converted the probability measure from a real-world measure to a risk-adjusted one produced by the constant market price of risk. The Sharpe ratio may potentially be modified using an adequate annuity quotation; however, as stated before annuity market values of longevity-related assets are insufficient.

For the **Risk Neutral** approach the main problem is the shortage of longevity-linked securities traded on the market, so payments associated with mortality cannot be replicated, that leads to the inability to estimate a single risk-adjusted probability measure. If the global market is arbitrage-free, there is at least one risk-neutral measure $\mathbb{Q}$ that can be used to calculate the fair price

The primary ambition of this study is to suggest a durability hedging and pricing mechanism in accordance with the Solvency II directive (1.4.2) in a stochastic continuous-time setting. In doing so are now presented some actuarial tools in order to understand the key elements needed. In Section 2 will be studied and compared continuous-time processes to project future mortality, and finally in Section 3 will be priced an S-forward with a new method consistent with the Solvency II directive, and it will be compared with the other three approaches listed above.

1.5 Basic elements of actuarial mathematics

The life expectancy of the insurer is the key element in our context, we will therefore dedicate this subsection to the presentation of this random variable (r.v.)

1.5.1 Life expentancy at birth

Consider a person aged $x = 0$ (at birth) and call T_0 his random life duration, calculated in years.
Be $F_0(t)$ the distribution function of T_0 $/$ $F_0(t) = Pr(T_0 \leq t)$

Being true the following assesments:
- T_0 determinations are positive real numbers
- the probability distribution of T_0 is continuous and has a density function; it takes values in $(0, \omega]$ with ω maximum age. So $\exists$ $f_0(t)$, non negative and normalized $/$ for $t < 0 \Rightarrow f_0(t) = 0$, called density function $/$ $\forall t \geqslant 0$ results $F_0(t) = \int_0^t f_0(u)du$

The plot of the density function is called death curve. Is now introduceable the survival process $S(t)$ $/$ $S(t) = P(T_0 > t) = 1 - F_0(t)$.

Knowing that for the distribution function:
$F_0(0) = 0$; $\lim_{t\to+\infty} F_0(t) = 1$,

it will be for the survival function:
$S(0) = 1$; $\lim_{t \to +\infty} S(t) = 0$

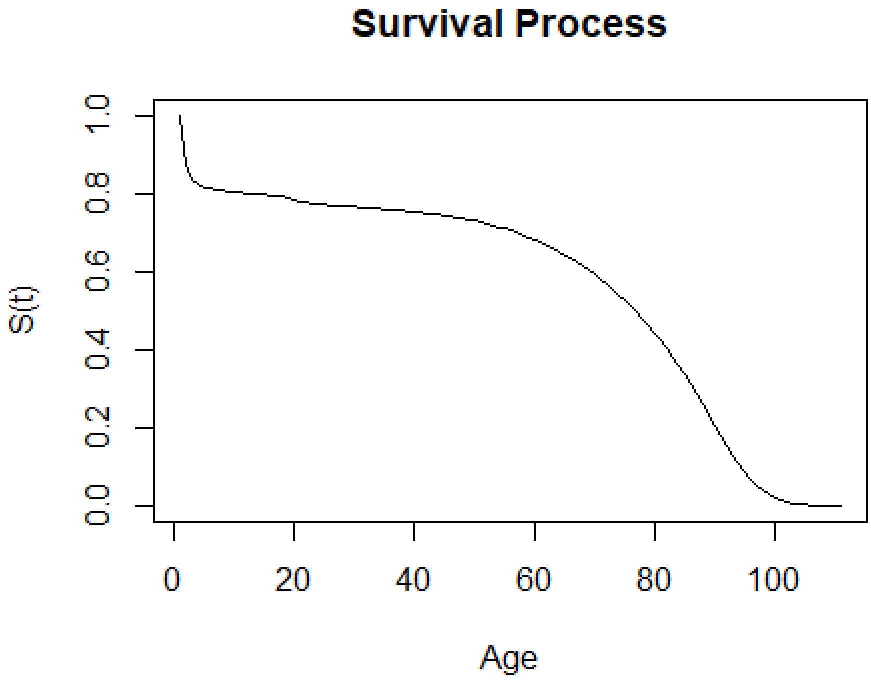


Figure 15: *The Survival Process*,
realized in R with the data of HMD for Italy

The above chart shows a plausible trend for the survival process. It should be noticed the initial decrement related to the infant mortality, while the inflection point shows the strong mortality in old ages

1.5.2 The residual life span at age x

In general we can now consider a person aged x, and put T_x his random life span. By definition is known that $T_x = (T_0 - x)|T_0 > x$. The probability distribution of T_x is identified by

$$
\begin{aligned}
F_x(t) &= P(T_x \leqslant t) = P(T_0 \leqslant x+t | T_0 > x) = \\
&= \frac{P(x < T_0 \leqslant x+t)}{P(T_0 > x)} = \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)} = \\
&= \frac{S(x) - S(x+t)}{S(x)} = 1 - \frac{S(x+t)}{S(x)}, t > 0
\end{aligned}
\tag{16}
$$

So that

$$
f_x(t) = -\frac{\frac{d}{dt} S(x+t)}{S(x)}, t > 0 \tag{17}
$$

1.5.3 Actuarial Notation

In the actuarial mathematics, is prevalent the use of the survival and death probability, stand respectively:

$$
{}_tq_x = F_x(t) \tag{18}
$$

$$
{}_tp_x = 1 -_t q_x \tag{19}
$$

So that

$$
{}_tq_x = 1 - \frac{S(x+t)}{S(x)} \tag{20}
$$

$$
{}_tp_x = \frac{S(x+t)}{S(x)} \tag{21}
$$

It can be immediately proved that:

$$
{}_tp_x =_\tau p_x \cdot_{t-\tau} p_{x+\tau}, 0 \leqslant \tau \leqslant t \tag{22}
$$

Is habit for $x = 0, 1, 2, 3, ...$, that:
${}_1p_x = p_x$,
${}_1q_x = q_x$,
and $\forall$ t integer from (22):

$$
{}_tp_x = p_x \cdot p_{x+1} \cdot ... \cdot p_{x+\tau-1} \tag{23}
$$

From the sequence q_x (or from p_x) it can be constructed the survival process $\forall\ x = 0, 1, 2, 3, ...$, s.t. $S(0) = 1$, for (20):

$$
S(x+1) = S(x)(1 - q_x) \tag{24}
$$

1.5.4 Intensity of mortality

Consider, for x and Δx positive and arbitrary, the following probability:

$$\begin{aligned} {}_{\Delta x}q_x = P(T_x \leqslant \Delta x) = P(T_0 \leqslant x + \Delta x | T_0 > x) = \\ = \frac{P(x < T_0 \leqslant x + \Delta x)}{P(T_0 > x)} = \frac{F_0(x + \Delta x) - F_0(x)}{1 - F_0(x)} = \\ = \frac{f_0 \Delta x}{S(x)} + o(\Delta x) \end{aligned} \tag{25}$$

where $o(\Delta x)$ is a higer order infinitesimal to Δx. Is put:

$$\mu(x) = \lim_{\Delta x \to 0} \frac{{}_{\Delta x}q_x}{\Delta x} \tag{26}$$

So that

$$\mu(x) = \frac{f_0(x)}{S(x)} \tag{27}$$

The function $\mu(x)$ defined above is known as intensity of mortality. His meaning is clear looking at (26) as we can write for Δx 'small':

$${}_{\Delta x}q_x \approx \mu(x) \Delta x \tag{28}$$

The death probability in $(x, x + \Delta x)$ (for a person aged x) is so proportional to the width Δx of the interval, for a coefficient $\mu(x)$, related generally to the age x. The (28) allows us to identify the dimension of $\mu(x)$, because known that ${}_{\Delta x}q_x$ is a probability (scale free) and Δx is a time, $\mu(x)$ has dimension the reciprocal of time, an intensity. From (17) and (27), we have also:

$$\mu(x) = -\frac{S'(x)}{S(x)} \tag{29}$$

i.e.

$$\mu(x) = -\frac{d}{dx} lnS(x) \tag{30}$$

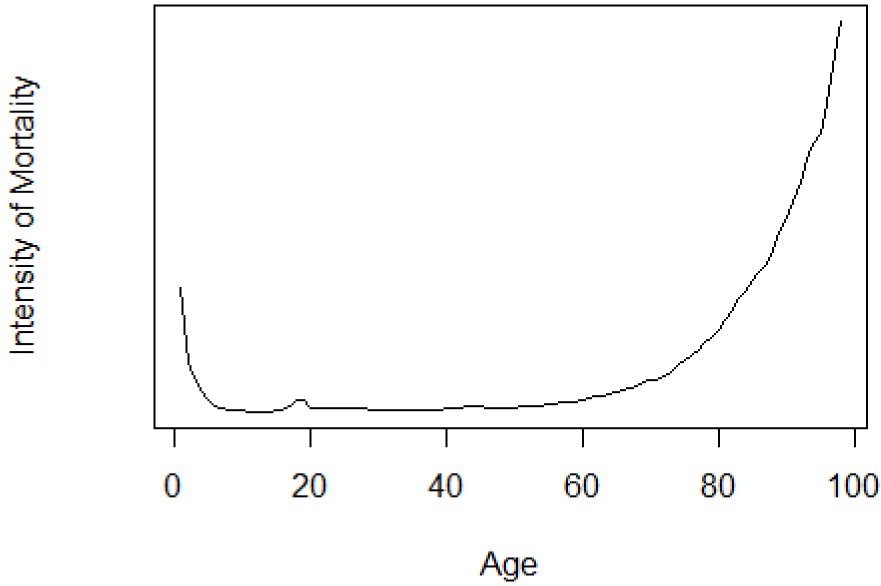


Figure 16: *The Intensity of Mortality*,
realized in R with the data of HMD for Italy

The chart above represent a plausible trend for $\mu(x)$. $\mu(x)$ can be interpreted in different ways. The (27) suggests $\mu(x)$ like density in x of the r.v. T_0, subject to $T_0 > x$. Also (29) suggests that the first derivative $S'(x)$ mesures the istantaneous speed of variation of the survival probability. A speed of variation has on the mortality an impact as strong as less the survival probability is (infact $\forall$ ages x, we study the "death risk" trough a mesure compared to $S(x)$, $\frac{S'(x)}{S(x)}$). From (27) and (29) we can make explicit the relationships $\mu(x)$ has with the others actuarial elements we have introduced before:

$$f_x(t) = -\frac{S'(x+t)}{S(x)} = \frac{S(x+t)}{S(x)}\frac{S'(x+t)}{S(x+t)} =_t p_x \mu(x+t) \tag{31}$$

$$S(x) = e^{-\int_0^x \mu(t)dt} \tag{32}$$

As $\lim_{x \to +\infty} S(x) = 0$, the intensity of mortality must prove the following statement:

$$\lim_{x \to +\infty} \int_0^x \mu(t)dt = +\infty \tag{33}$$

For actuarial models, as we will see soon, the intensity of mortality is ever-increasing, so the (33) will be satisfied. Once found $S(x)$, we're able to write other functions related to the probability description of the random life duration. Is also interesting express these functions in terms of $\mu(x)$

$$p_x = \frac{S(x+1)}{S(x)} = e^{-\int_0^1 \mu(x+t)dt} \tag{34}$$

$$q_x = 1 - e^{-\int_0^1 \mu(x+t)dt} \tag{35}$$

This work aims to pricing longevity-linked securities, to make this, a central point is clearly linked to how to model future mortality. Will be developed in this context an approach that will follow the modelling of $\mu(x)$ with affine processes [1], as we shall see later in section 2. A first approach in this sense is starting from the classic actuarial models for the intensity of mortality. The function intensity of mortality must be modelled with logic biological assuptions. The Gompertz model (1825) is based on the assumption that $\mu(x)$ increases in an interval Δx in proportion to Δx and to the starting $\mu(x)$; i.e:

$$\Delta\mu(x) = \beta\mu(x)\Delta x + o(\Delta x), \beta > 0 \tag{36}$$

then dividing by Δx and for $\Delta x \longrightarrow 0^+$, you have

$$\frac{d}{dx}\mu(x) = \beta\mu(x) \tag{37}$$

therefore integrating:

$$\mu(x) = \alpha e^{\beta x}, \alpha > 0 \tag{38}$$

A simple generalise of the Gompertz law above is the Makeham's law (1860). In this one is explicit the influction in the mortality af accidentals causes (independents from aging), expressed with an adding costant γ

$$\mu(x) = \gamma + \alpha e^{\beta x}, \alpha, \beta > 0; \gamma \geqslant 0 \tag{39}$$

[1] see in Appendix B

Putting $\gamma = 0$ we have the Gompertz's law, for $\beta = 0$ the Dormoy's, with an unrealistic costant intensity. From (32) we can obtain the following expression of the survival process:

$$S(x) = ks^x g^{c^x} \tag{40}$$

with $s = e^{-\gamma}$, $g = e^{-\frac{\alpha}{\beta}}$ $c = e^{\beta}$, $k = \frac{1}{g}$.

Comparing the survival processes of Makeham and Gompertz with the survival tables built with a lot of statistical observation, we can see that both laws give a good fitting in wide age intervals. Wider with Makeham (from 25-30 years to 95). In both laws however, the monotonic increasing trend of the intensity, doesn't reflect the mortality in infant age (but it is not important for insurance applications), neither the increasing of the mortality for accidental causes in 18-25 years old that emerge in empirical tables. The Lazarus's law (1867) generalised the Makeham's, with an another exponential adding term, type "negative Gompertz", infact this law is also called "double exponential", his need is to reflect the mortality also in infant age.

$$\mu(x) = \alpha_1 e^{-\beta_1 x} + \gamma + \alpha_2 e^{\beta_2 x}, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0; \gamma \geqslant 0 \tag{41}$$

Last, the Thiele's law (1867), that reflects also the trend of mortality in the young age:

$$\begin{aligned} \mu(x) = \alpha_1 e^{-\beta_1 x} + \alpha_3 e^{-0.5\beta_3 (x-\gamma)^2} + \alpha_2 e^{\beta_2 x}, \\ \alpha_1, \alpha_2, \beta_1, \beta_2 > 0; \gamma, \alpha_3, \beta_3 \geqslant 0 \end{aligned} \tag{42}$$

This law recognize in the first term the infant mortality (like Lazarus), the second terms (gaussian trend), is related to accidental mortality, the third (Gompertz type) explain the elderly mortality. Is interesting observing that this law in three terms has been proposed in 1980 by Helingman-Pollard, that has the same structure and the same purpose, but it is constructed for the evolution of the "Odds"

$$\frac{qx}{px} = A^{(x+B)^C} + De^{-E(lnx - lnF)^2} + GH^x \tag{43}$$

1.5.5 Aggravated Risk

A natural consequence of the thread we are following, is to wonder if the modelling of future mortality cannot be adequate ex-ante, taking into account possible future variations in the trend of the intensity of mortality. In the actuarial field, an approach often used to express a "personalization" of the survival probability valuation, consist in the use of affordable models aimed to distort the mortality from the one of the table or related to a "standard" intensity. In this way is possible to express aggravation or reduction of mortality. In the life insurance business, the outcome of the medical examination and the statements made by the person at the time of entry into insurance may reveal aggravations due to the health conditions or the particular professional activity of the person. Such aggravations can be quantified through the models we will now present, starting from the application of the same to a generic function of intensity of mortality. The most commonly adopted assumptions on aggravation of mortality, for an insurer aged x at the entry are followed described, note that we use the common actuarial application that refears to an aggravation: the longevity is a reduction, but the construction remains the same, and we adopt this notation. We will indicate μ like the standard intensity of mortality, and $\mu^{()}$ like the aggravated one. With $\eta^{()}$ we denote the aggravation function, i.e. the difference between the standard and the aggravated intensity of mortality (see Pitacco [26]). Is it natural to assume that between the two intensities, there is ∀x and t⩾0 a linear relation such that:

$$\mu^{()}(x,t) = (1+\gamma)\mu(x,t) + \delta\mu(x) \tag{44}$$

Note that for $\delta = 0$ we obtain the moltiplicative model that we can indicate as $\mu^{(m)}$:

$$\mu^{(m)}(x,t) = (1+\gamma)\mu(x,t) \tag{45}$$

i.e.

$$\eta^{(m)}(t) = \gamma\mu(x,t) \tag{46}$$

The aggravation results costant, so at the increasing of the "standard" μ increase the aggravation. In the same way we can obtain the additive model with $\gamma = 0$:

$$\mu^{(a)}(x,t) = \mu(x,t) + \delta\mu(x) \tag{47}$$

i.e.

$$\eta^{(a)}(t) = \delta\mu(x,t) \tag{48}$$

2 Old and new methods for managing future modelling

2.1 How precise can we be?

Companies are constantly seeking to address the issue and implement new ideas to cover longevity risk, particularly with the Solvency II Directive. The first task is to properly model the evolution of longevity in order to make precise predictions for the future. The regulator needs the incorporation of mortality risk analysis into stochastic valuation models, and we know from the literature that stochastic models should be more reliably to quantify mortality (see Cairns et al. [27]), provided that such models are the better way to capture the complexity implicit in the problem. Researchers have suggested a number of stochastic mortality models. The discrete time model developed by Lee and Carter [28] is the most commonly used mathematical model for mortality in the current literature (time series technique). Milevsky and Promislow [29] were the first to suggest a stochastic force of mortality model, which is based on the use of continuous-time stochastic processes for explaining the force of mortality. And since, many other stochastic models based on the comparison of mortality and interest rates have been suggested (Dahl[30], Biffis [31], Denuit and Devolder [32], Luciano and Vigna [33], Schrager [34], Zeddouk [35]). Some of the proposed models are based on the natural analogy between mortality intensities and interest rates (or default rates). The survival function for a person can be given in a closed form by using affine processes for intensity of mortality, which is incredibly helpful for pricing mortality/longevity-linked securities.

2.1.1 Discrete time models

Initiatives in research and development have increasingly concentrated on stochastic mortality models in order to help forecast longevity risk. We will investigate now about discrete time models, based on extrapolative methods. Two families of models well known in the literature and most used in actuarial evaluations, are the Lee-Carter's and extensions and the Cairns-Blake-Dowd's and extensions. The Lee-Carter model was the first one to consider increased life expectancy patterns in age mortality dynamics, and it has been used for stochastic projections of the United States' Social Security system as well as

other elements of the United States' federal budget.
Lee and Carter's purpose is to describe the age-period surface of log-mortality rates in terms of vectors α and β along the age dimension and k along the time dimension, in particular α describes the behavior of mortality with varying age, β describes for each age how mortality reacts to the variation of k, that is the index of change in mortality over time, plus an error term $\epsilon_{x,t}$, identically distributed independent errors with standardized normal distribution $N(0, \sigma_\epsilon^2)$

$$ln(m_{x,t}) = \alpha_x + \beta_x k_t + \epsilon_{x,t} \tag{49}$$

The parameterization can be done with the constraints adopted by the autors $\sum_t k_t = 0$ and $\sum_x \beta_x = 1$, that immediately leads us to conclude that the parameter α_x is the average over time of the age profile. The estimated parameters k are then modelled and projected as a stochastic time series using ARIMA models.
The Lee-Carter model implicitly assumes that random errors $\epsilon_{x,t}$ are homoschedastic (same variance with respect to age), unrealistic assumptions for high ages, where there is a increased variability of mortality due to the small number of deaths. The solution was proposed by Brouhns et al. (2002)[36]: central mortality rates modeled using the Lee-Carter model: $ln(m_{x,t}) = \alpha_x + \beta_x k_t$, with deaths that follow a Poisson distribution: $D_x(t) \sim Poisson(E_x(t) m_x(t))$. The estimation of Lee-Carter parameters is computed as follow:

- Definition of an objective function to minimize (Least Squares):

$$O_{LS}(\alpha, \beta, k) = \sum_{x=x_1}^{x_m} \sum_{t=t_1}^{t_n} (ln\hat{m}_{x,t} - \alpha_x - \beta_x k_t)^2 \tag{50}$$

  with $\hat{m}_{x,t}$ observed rates

- Singular value decomposition:

$$\frac{\partial}{\partial \alpha_x} O_{LS} = 0 \tag{51}$$

  i.e.

$$\sum_{t=t_1}^{t_n} ln\hat{m}_{x,t} = (t_n - t_1 + 1)\alpha_x + \beta_x \sum_{t=t_1}^{t_n} k_t \tag{52}$$

that s.t. $\sum_t k_t = 0$ we obtain

$$\hat{\alpha}_x = \frac{1}{(t_n - t_1 + 1)} \sum_{t=t_1}^{t_n} ln\hat{m}_{x,t} \tag{53}$$

- Parameters β and k are obtained by the SVD of the first term of the matrix $ln\hat{m}_{x,t} - \hat{\alpha}_x = Z$, β and k minimize the following:

$$O_{LS}(\beta, k) = \sum_{x=x_1}^{x_m} \sum_{t=t_1}^{t_n} (z_{xt} - \beta_x k_t)^2 \tag{54}$$

  Is needed to find the eigenvectors and eigenvalues of the matrix Z defined as previous.

- Lastly the k parameters are calibrated on the observed death ditribution by age

The Lee-Carter model requires constraints on β and k for the calibration, otherwise we will have problems of identification of parameters, furthermore the β parameter may be negative for some ages, indicating that mortality for those ages tends to increase while decreasing at different ages.

Empirical analysis of mortality data suggest that the natural logarithm of odds, $ln\frac{q_x(t)}{p_x(t)}$ takes a linear form with respect to age x over a time period of t years. Cairns et al.[37] proposed the following model with two time terms:

$$ln\frac{q_x(t)}{p_x(t)} = k_t^{[1]} + k_t^{[2]}x \tag{55}$$

i.e.

$$q_x(t) = \frac{exp(k_t^{[1]} + k_t^{[2]}x)}{1 + exp(k_t^{[1]} + k_t^{[2]}x)} \tag{56}$$

sometimes x is replaced by $x - \overline{x}$. The log-odds could be written as $logitq_x(t)$. $k_t^{[1]}$ and $k_t^{[2]}$ are two stochastic processes that form a bivariate time series and govern the projection of mortality rates. The Cairns-Blake-Dowd model has not parameter identification problems, does not requires constraits. Generally $k_t^{[1]}$ decreases in time like the Lee-Carter's, showing how the mortality rates decrease in time for all ages x. If during the data observation period mortality increases are higher at youth age than at old age, then $k_t^{[2]}$ increases over time.

Compared to the Lee-Carter model, the Cairns-Blake-Dowd model shows changes in mortality rates that are not fully age-related. The estimation of Cairn-Blake-Dowd parameters is computed as follow:

- Are used Least Squares, with the following regression model:

$$ln\frac{\hat{q}_x(t)}{\hat{p}_x(t)} = k_t^{[1]} + k_t^{[2]}x + \epsilon_x(t) \tag{57}$$

  with error term $\epsilon_{x,t}$, identically distributed independent errors with standardized normal distribution $N(0, \sigma_\epsilon^2)$

- Definition of an objective function to minimize that gives the estimation of the time parameters:

$$O_t(k) = \sum_{x=x_1}^{x_m} (ln\frac{\hat{q}_x(t)}{\hat{p}_x(t)} - k_t^{[1]} - k_t^{[2]}x)^2 \tag{58}$$

Despite the fact that data are generally presented at discrete periods (yearly), other models have taken into consideration the evolution of death rates over time. With the requirements of Solvency II for assessing mortality risk, insurance companies have become particularly active in continuous-time stochastic mortality models. and a vast array of continuous-time stochastic models have been studied by several authors.

2.1.2 Continuous time models, mean reverting or not?

Our approach concentrates on mortality and measuring individuals' survival functions using one-factor short-rate models, which are utilized in finance to explain variations in interest rate term structure. Milevsky used tools and techniques established in interest rate analysis to model the force of mortality, leveraging analogies between mortality and interest rate dynamics. Non-mean reversion models, according to the research, are more suited for mortality prediction than mean reversion models with a defined long-term endpoint. We investigate in this subsection the effect of adding a time-dependent long-term mean reversion level to mean-reverting systems. The model's eventual mean-reverting feature is critical in this setting. The question of mean reversion models is motivated by the parallel between interest rates and the force of mortality models. The mean reversion idea is simple and incorporates long-term convergence to an equilibrium level with random noise. Luciano and Vigna [33] argue that non-mean-reverting affine processes are superior at simulating the force of mortality, based these results on fixed-target reversion models. This comparison is revisited by Zeddouk [35] by evaluating both fixed-target mean-reverting models and moving-target mean-reverting models. Predictably, a fixed-time endpoint is problematic for depicting the development of time on mortality intensity, which rises with age. Examining a mean-reverting model with a rising target, particularly one with an exponential target compatible with the traditional Gompertz model (38), is indeed more interesting. In this subsection we model the force of mortality $\mu_x(t)$ (26) as a continuous stochastic process:

Given $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $\mathcal{F}_t$, the process $\mu_x(t, \omega)$ can be defined, with $\omega \in (\Omega, \mathcal{F}, \mathbb{P})$. Defined as previous $\mu_x(t, \omega)$, from (30) we know that:

$$S(x, t, T, \omega) = e^{-\int_t^T \mu_x(u,\omega)du} \tag{59}$$

So that in t the survival probability is computed as the expected survival process conditioned to the information $\mathcal{F}_t$:

$${}_{T-t}p_{x+t} = \mathbb{E}^{\mathbb{P}}[S(x, t, T, \omega)|\mathcal{F}_t] \tag{60}$$

For Luciano and Vigna [33], continuous mean reversion models are unsuitable for describing individual mortality; consequently, non-mean reversion models perform better. Therefore, we will consider three models, one non-mean-reverting, one mean-reverting with fixed target (Vasicek), one mean-reverting with a modified moving target (Hull-White). The significance of using affine models to describe $\mu_x(t)$ is the efficiency with which an explicit formula for the survival probability (60) can be found. As a result, for affine processes under technical constraints (see Duffie et al.[38]), the classical stochastic process characteristics enable us to write:

$${}_{T-t}p_{x+t} = e^{A(t,T)-B(t,T)\mu_x(t)} \tag{61}$$

where $A(t, T)$ and $B(t, T)$ are deterministic functions and solutions of the Riccati equation that may be found numerically and, in certain instances, analytically (Duffie et al.[39]).

Non-mean-reverting model

Is given the following Stochastic Differential Equation (SDE) [2] for the intensity of mortality:

$$d\mu_x(t) = \alpha\mu_x(t)dt + \sigma dZ_t \tag{62}$$

$\alpha > 0$, $\sigma > 0$ and Z_t standard Brownian motion (Wiener Process)[3]. Because the model is Gaussian, we can determine the term of the survival probability by analyzing its distribution. We may alternatively employ the affine structure, which yields the following system of ordinary differential equations for $A(t,T)$ and $B(t,T)$ (see Luciano and Vigna, 2015 [40]).

$$\begin{cases} \frac{dA(t,T)}{dt} & = \frac{1}{2}\sigma^2 B(t,T)^2 \\ \frac{dB(t,T)}{dt} & = \alpha B(t,T) - 1 \\ A(T,T) & = B(T,T) = 0 \end{cases} \tag{63}$$

When resolving the system (63), we discover:

$$\begin{cases} A(t,T) & = \frac{\sigma^2(T-t)}{2\alpha^2} + \frac{\sigma^2}{4\alpha^3}e^{2\alpha(T-t)} - \frac{\sigma^2}{\alpha^3}e^{\alpha(T-t)} + \frac{3\sigma^2}{4\alpha^3} \\ B(t,T) & = \frac{1}{\alpha}(1 - e^{\alpha(T-t)}) \end{cases} \tag{64}$$

The solution of the Stochastic Differential Equation (62) is:

$$\mu_x(t) = \mu_x(u)e^{\alpha(t-u)} + \sigma\int_u^t e^{\alpha(t-s)}dZ_s \tag{65}$$

$\forall$ u,t ／ (0⩽u⩽t⩽T).

The intensity of mortality has positive probability of being negative:

$$P(\mu_x(t) < 0|F_u) = \phi(-\frac{\mu_x(u)e^{\alpha(t-u)}}{\sigma\sqrt{\frac{e^{2\alpha(t-u)}-1}{2\alpha}}}) \tag{66}$$

Where ϕ is the distribution function of a standard normal r.v.

[2] see Appendix A

[3] see Appendix A

Mean-reverting model

We decide to study in this section the classical Vasicek [41] model, retracing the Luciano and Vigna's work and the Zeddouk's. With Vasicek (1977) is assumed that the intensity of mortality under $\mathbb{P}$ is described by an Ornstein-Uhlenbeck process:

$$d\mu_x(t) = \alpha(\gamma - \mu_x(t))dt + \sigma dZ_t \tag{67}$$

$\gamma > 0$, $\alpha > 0$, $\sigma > 0$, Z_t the Wiener process. The drift term shows a mean reversion to γ, with speed α. Integrating the equation (67) $\forall\ u \leqslant t$:

$$\mu_x(t) = \mu_x(u)e^{\alpha(t-u)} + \gamma(1 - e^{-\alpha(t-u)}) + \sigma \int_u^t e^{\alpha(t-s)}dZ_s \tag{68}$$

The probability distribution of $\mu_x(t)$ in s with s$\geqslant$t is a Normal r.v. with mean and variance under the probability measure $\mathbb{P}$:

$$\mathbb{E}_t^{\mathbb{P}}[\mu_x(s)] = \gamma + (\mu_x(t) - \gamma)e^{-\alpha(s-t)} \tag{69}$$

we can easily see that:

$$for \begin{cases} s \longrightarrow t & \Rightarrow \mathbb{E}[\mu_x(s)] \longrightarrow \mu_x(t) \\ s \longrightarrow \infty & \Rightarrow \mathbb{E}[\mu_x(s)] \longrightarrow \gamma \end{cases} \tag{70}$$

$$\mathrm{Var}_t^{\mathbb{P}}[\mu_x(s)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(s-t)}) \tag{71}$$

$$for \begin{cases} s \longrightarrow t & \Rightarrow \mathrm{Var}[\mu_x(s)] \longrightarrow 0 \\ s \longrightarrow \infty & \Rightarrow \mathrm{Var}[\mu_x(s)] \longrightarrow \frac{\sigma^2}{2\alpha} \end{cases} \tag{72}$$

Where from $s \longrightarrow \infty$ we see that the variance increases proportionally at the dispersion term (σ multiply the Brownian), and decreases proportionally at α that drives at the convergence in mean.

The Vasicek model is an affine model so we can write analitically the survival probability as:

$${}_{T-t}p_{x+t} = e^{A(t,T) - B(t,T)\mu_x(t)} \tag{73}$$

where $A(t, T)$ and $B(t, T)$ are given from:

$$\begin{cases} A(t,T) & = exp[(\gamma - \frac{\sigma^2}{2\alpha^2})(B(t,T) - T + t) - \frac{\sigma^2}{4\alpha}B(t,T)^2] \\ B(t,T) & = \frac{1}{\alpha}(1 - e^{-\alpha(T-t)}) \end{cases} \tag{74}$$

Luciano and Vigna in their "Non mean reverting affine processes for stochastic mortality", show how the property of mean reversion is insuitable for the modelling of the force of mortality, comparing the two models we presented above. Infact in this work the non-mean reverting model overperform the Vasicek. Based on this work the solution may be reject the mean reverting property. Zeddouk in her "Mean reversion in stochastic mortality: why and how?" added this lack of performing in the fixed target γ, in fact the intensity of mortality, unlinke the interest rate models, has a growing trend with age and also with time in (67). So as a result, appears necessary to re-evaluate the performance of the mean-reverting models with an increasing target. In this context, a natural candidate is to employ an exponential form as the rising target, as in the classical Gompertz model (38).

Mean-reverting moving target model

For the moving target category we analyze the Hull and White model (1990)[42], a time-dependent model, extension of the Vasicek model, theorically needed for an exact fit to the currently-observed yield curve that led Hull and White to introduce a time-varying parameter, corrisponding to the Vasicek's γ, chose as a deterministic funcion of time $\vartheta(t)$. This model implies a Gaussian distribution for the process, and the survival probability is analytically determinable.

$$d\mu_x(t) = (\vartheta(t) - \alpha\mu_x(t))dt + \sigma dZ_t \tag{75}$$

$\alpha > 0$, $\sigma > 0$, Z_t the Wiener process and $\vartheta(t)$ a deterministic function, modelled with the Gompertz's law (38), i.e.:

$$\vartheta(t) = \kappa e^{\beta t} \tag{76}$$

The Hull and White model is in this context as follows:

$$d\mu_x(t) = \alpha(\frac{\kappa}{\alpha}e^{\beta t} - \mu_x(t))dt + \sigma dZ_t \tag{77}$$

If we consider $\frac{\kappa}{\alpha}$ as the starting point $\mu_x(0)$ the model could be written as:

$$d\mu_x(t) = \alpha(\mu_x(0)e^{\beta t} - \mu_x(t))dt + \sigma dZ_t \tag{78}$$

The Hull and White is Gaussian, the survival probability can be found in closed form. Integrating the equation (77) $\forall\ u \leqslant t$:

$$\mu_x(t) = \mu_x(u)e^{-\alpha(t-u)} + \frac{\kappa}{\alpha+\beta}(e^{\beta t} - e^{\beta u - \alpha(t-u)}) + \sigma e^{-\alpha t}\int_u^t e^{\alpha s} dZ_s \tag{79}$$

The probability distribution of $\mu_x(t)$ in s is a Normal r.v. with mean and variance under the probability measure $\mathbb{P}$:

$$\begin{cases} \mathbb{E}^{\mathbb{P}}[\mu_x(t)|F_s] = (\mu_x(s)e^{-\alpha(t-s)} + \frac{\kappa}{\alpha+\beta}(e^{\beta t} - e^{\beta s-\alpha(t-s)}) \\ \text{Var}^{\mathbb{P}}[\mu_x(t)|F_s] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(t-s)}) \end{cases} \tag{80}$$

Like the Ornestein-Uhlenbeck process, $\forall t$ the force of mortality $\mu_x(t)$ can be negative with a probability given by:

$$\phi(-\frac{(\mu_x(s)e^{-\alpha(t-s)} + \frac{\kappa}{\alpha+\beta}(e^{\beta t} - e^{\beta s-\alpha(t-s)})}{\sqrt{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(t-s)})}} \tag{81}$$

Furthermore with the boundary condition $A(T,T) = B(T,T) = 0$ we can solve the system of ordinary differential equations:

$$\begin{cases} \frac{dA(t,T)}{dt} & = B(t,T)\vartheta(t) - \frac{1}{2}\sigma^2 B(t,T)^2 \\ \frac{dB(t,T)}{dt} & = \alpha B(t,T) - 1 \end{cases} \tag{82}$$

i.e.

$$\begin{cases} A(t,T) = & \frac{\kappa}{\alpha}[e^{-\alpha T}\frac{e^{(\beta+\alpha)T} - e^{(\beta+\alpha)t}}{\beta+\alpha} - \frac{e^{\beta T} - e^{\beta t}}{\beta}] \\ & -\frac{\sigma^2}{2\alpha^2}[\frac{1}{\alpha}(1 - e^{-\alpha(T-t)}) - T + t] - \frac{\sigma^2}{2\alpha^3}(1 - e^{-\alpha(T-t)})^2 \\ B(t,T) = & \frac{1}{\alpha}(1 - e^{-\alpha(T-t)}) \end{cases} \tag{83}$$

2.2 Models Comparison

In this subsection, we use statistical strategies to assess the efficiency of the mentioned models in predicting mortality. The models are calibrated using past and projected mortality data. Following that, we compare the mortality trends for the various models. Lastly, we employ the models to forecast mortality.

2.2.1 Calibration

We calibrate the models to a few cohorts of the Italian population and evaluate their ability to forecast mortality using various statistical techniques.

In particular we consider three different cohorts of individuals, born in 1900, 1915 and 1955; with started point at age $x_0 = 50$, using historical data from Human Mortality Database for the cohorts of 1900 and 1915, and the projected table built by ANIA for the generation of 1955. We consider unisex annual death rates.
Be $p_1, p_2, ..., p_n$ the survival probability for a year, the probability that a member aged x_0, arrives at age $x_0 + t$ is:

$$_t\hat{p}_{x_0} = \prod_{j=x_0}^{x_0+t-1} p_j \tag{84}$$

Be $_{t_k}p_x(\underline{\lambda})$ the objective function to calibrate defined in (60), with $\underline{\lambda}$ the parameters' vector of the model to be calibrated. $\underline{\lambda}$ are estimated using Least Squares, the easiest way to procede, minimizing the errors between the survival probabilities predicted by the stochastic models and the observed one.

$$O_{LS} = \min_{\underline{\lambda}} \sum_{k=0}^{n} ({}_kp_x(\underline{\lambda}) -_k \hat{p}_x)^2 \tag{85}$$

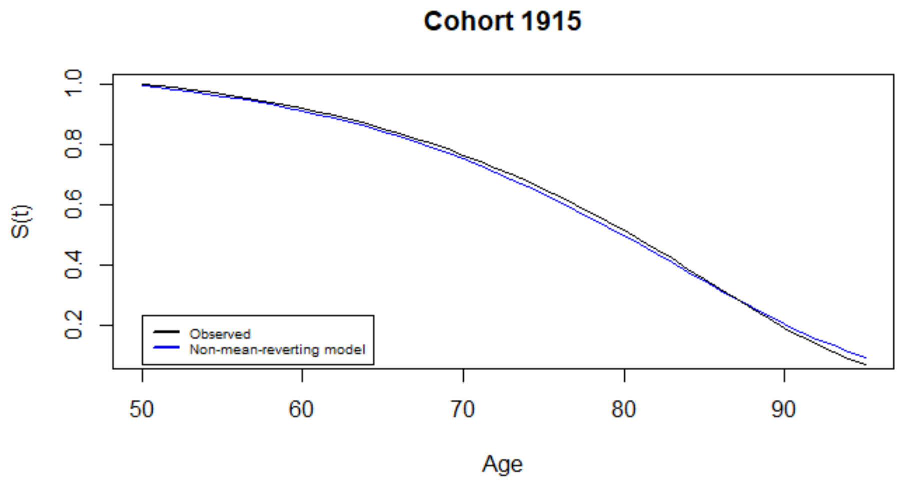
It must be solved a non linear system with the generalised Newton-Raphson method in R, but the output is sensitive to the starting values needed, for the optimal choice are randomly generated different starting points (see Zeddouk [35]). Using the data and the method described above, with starting point $\mu_x(0) = -ln(p_{50})$, are estimated the parameters of the three models for the three different cohorts of individuals:

Non-mean-reverting model

	1900	1915	1955
$\mu_{50}(0)$	0.007064898	0.005454851	0.002398374
error	0.005914428	0.003332871	0.062757120
α	0.075985339	0.0770249700	0.074757481
σ	0.000000126	0.000000132	0.000000217

Table 3: Estimated parameters for the survival process in the non-mean-reverting process

The values are consistent for this model. The α parameter has a declining tendency. The σ values are quite little, leading to the evidence that the volatility in mortality is considerably smaller than in finance. It must be observed that residuals are much more significant for the generation of 1955. These results are in the following charts, where is clear that the fit is quite good for the 1900 and 1915 cohorts, while is not accetable for the 1955 generation.



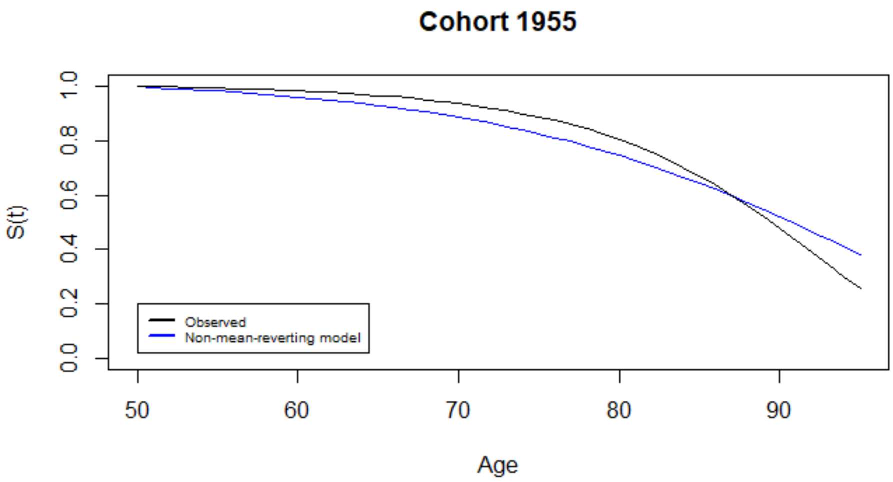


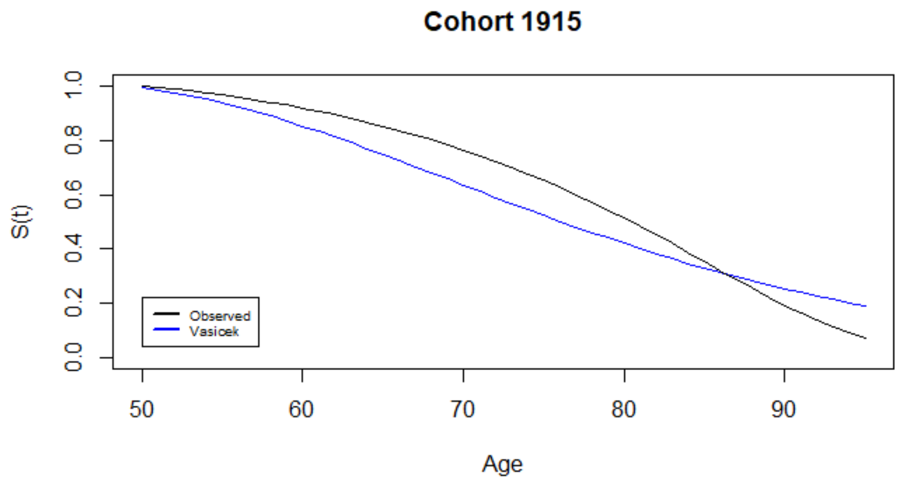
Figure 17: Survival functions, 1900,1915,1955 generation for non-mean-reverting model

Vasicek

	1900	1915	1955
$\mu_{50}(0)$	0.007064898	0.005454851	0.002398374
error	0.335919797	0.343592874	0.362467920
α	0.023562120	0.019998012	0.011913990
γ	0.002235632	0.001876597	0.000873345
σ	0.000027929	0.000010321	0.000334569

Table 4: Estimated parameters for the survival process in the Vasicek model

In this case the residuals are huge for all the cohorts of individuals. The fixed target γ highly decreases through the generations as expected. As can be seen also in this model the σ paremeter is very low. The fitting of the Vasicek model is not adequate, confirming the Luciano and Vigna's result: non-mean-reverting model are better than mean-reverting-model to project the mortality. As previous, the charts:



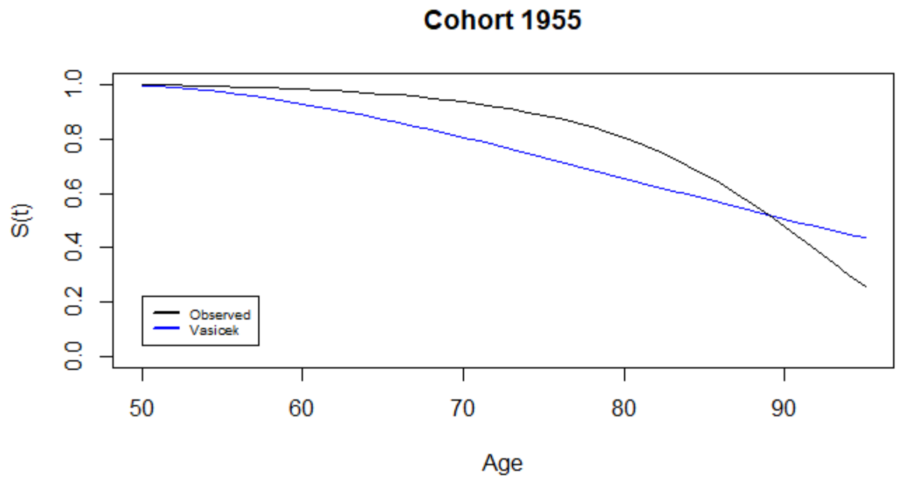


Figure 18: Survival functions, 1900,1915,1955 generation for Vasicek model

Hull and White

	1900	1915	1955
$\mu_{50}(0)$	0.007064898	0.005454851	0.002398374
error	0.000020023	0.000053323	0.0009247920
α	0.634769826	0.610482593	0.2021555423
κ	0.004484584	0.003330091	0.0004848446
β	0.083061184	0.082626391	0.0983905356
σ	0.015988143	0.000244231	0.0068082860

Table 5: Estimated parameters for the survival process in the Hull and White model

The residuals are very small in this model for all the cohorts of individuals. The parameter α is more significant in the older generations, because the target is lowering, that is the same reason that leads the huge decrease for the cohort of 1955 (10^{-1}) of the κ, and less intuitively is the same for the β parameter that increases in the younger generation. The σ parameter is very low for all the generations as the others models. The fitting is almost perfect, confirming the Zeddouk's result for the Belgians'cohorts: mean-reverting-moving target models are the best way to project the mortality. As previous, the charts:

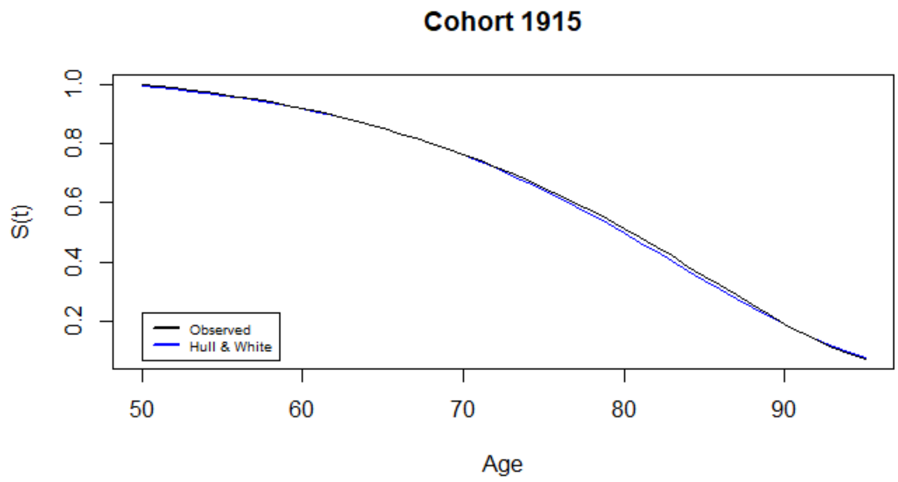


Figure 19: Survival functions, 1900,1915,1955 generation for Hull and White model

Finally, a comparison betwenn the calibration errors, the discrepancies between observed and theoretical survival probability for all generations. In the table below are depicted the errors for each model and cohort:

	1900	1915	1955
Non-mean-reverting	0.005914428	0.003332871	0.062757120
Vasicek	0.335919797	0.343592874	0.362467920
Hull and White	0.000090023	0.000053323	0.008357920

Table 6: Calibration errors

The significant error of the Vasicek model confirms the Luciano and Vigna's result in comparing non-mean-reverting and mean-reverting processes; the Vasicek is not adequate to depict $\mu_x()$. The non-mean-reverting errors are satisfyng, but not as good as the implementation of the Hull and White model whit his proprierties, confirming the Zeddouk's result. It can be usefull also to calcute the BIC, a seltection method that implies a good fit, without overfitting problems. In the following table the BIC results for the models:

	1900	1915	1955
Non-mean-reverting	-446.5632	-453.2042	-289.8423
Vasicek	-213.9432	-233.4323	-198.0392
Hull and White	-690.2341	-499.3241	-399.321

Table 7: BIC for all cohorts

The more the value of the BIC is low the more the model is appropriate, remembering the definition

$$BIC = nln(\frac{\epsilon}{n}) + pln(n) \tag{86}$$

with n the number of observations, p the number of parameters and ϵ the residuals of the Least Squares. Is again clear that the result of the Hull and White model is the best. Based on this evidence we keep the Hull and White model for the projection of the future mortality and for the pricing of the longevity-linked securities

2.3 Mortality Projection

Are compared now the mean of the force of mortality, from $\mu_x(50)$ to $\mu_x(95)$, plotting the force of mortality for 45 years, to study the path of the logevity through different cohorts, using the non-mean-reverting model and the HW model. Are used the parameter stimated in the previous subsection and a number of 100.000 simulations.

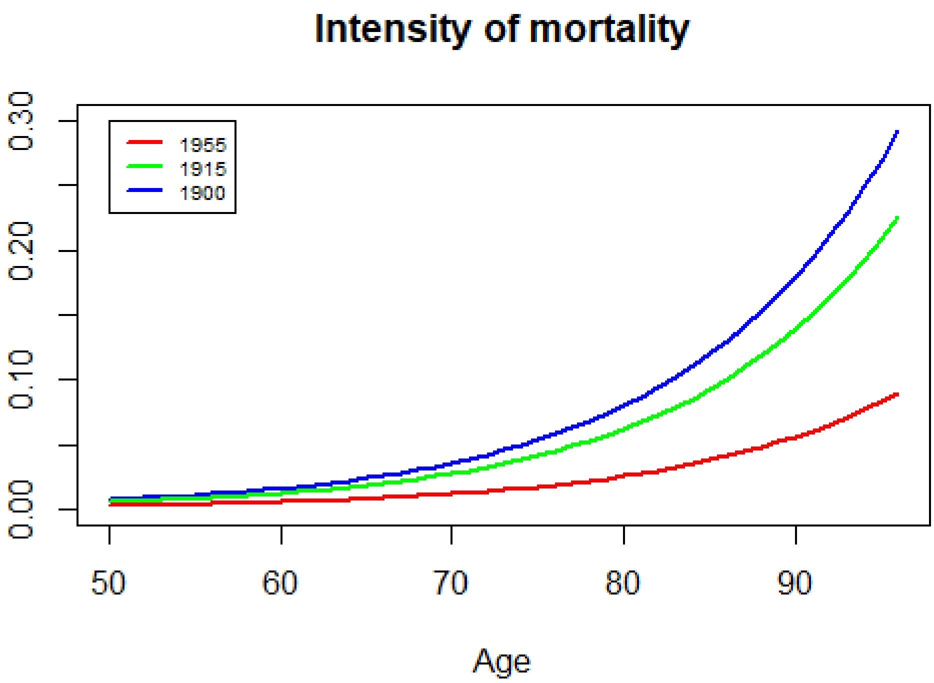


Figure 20: Mean of μ_x for the three cohorts in Non-mean-reverting model

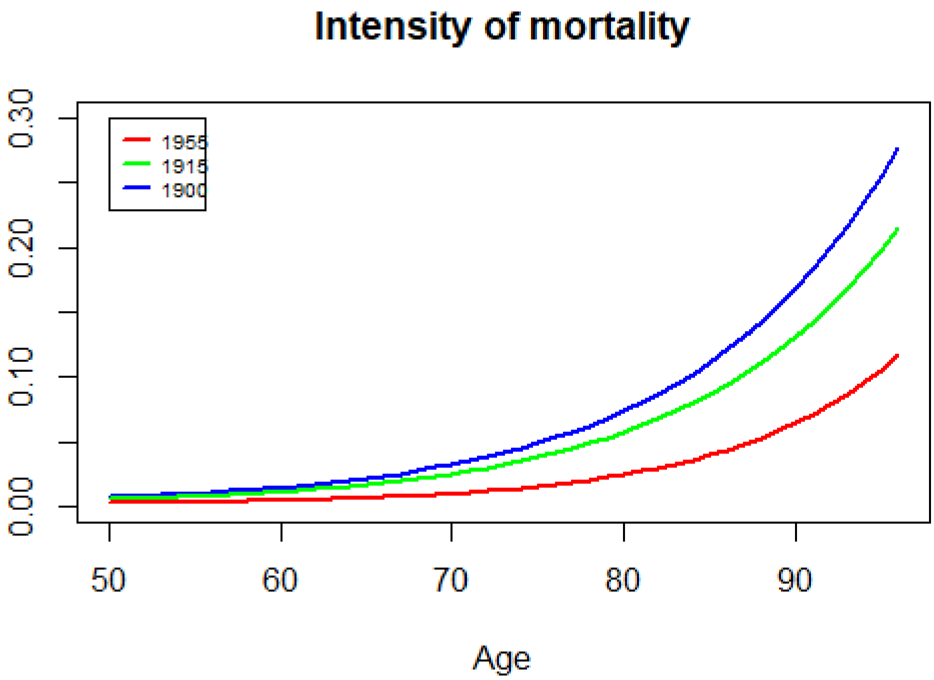


Figure 21: Mean of μ_x for the three cohorts in Hull and White model

For both models, older is the generation, higher is the mean of $\mu_x(t)$. Is possible to observe that for the mean reverting model, the mean of $\mu_x(t)$ seems increasing more rapidly with age.

Have been used three continuous-time processes to represent the trend of mortality, a way of operating very much in line with the Solvency II directive. These three different types of stochastic processes differed in their drift and diffusion parts, and their survival probabilities were given in closed form and examined analytically thanks to their affines properties. The intensity process has been calibrated with the Italian population using historical data for previous generations (from HMD) and a projected mortality table for the 1955 cohort. For each model has been provided the $S_{50}(t)$ function for 45 years in the three cohort analyzed and have been studied the residuals between observed series and simulated series. As a consequence, has been showed that adding a time-dependent long-term mean to the processes analyzed appears to be more suited for describing individual death intensity, fulfilling the majority of the requirements for a mortality model. Lastly for the non-mean reverting process and for the Hull and White has been projected the best estimate of the future mortality. Has been found that, while the mean reversion feature is unsuitable for mortality when using a constant target(Vasicek), it is ideal when using a variable target connected to the increase in mortality with age. An intriguing topic for future research would be to examine a mean reverting moving target process with a jump component that could follow a compound poisson process , with a given intensity λ related to the historical evidences of improved mortality (scientific and medical discoveries) and worsering trends (pandemics)

3 Pricing of longevity derivatives

As life expectancy increases, annuity providers are increasingly facing longevity risks. In order to hedge this risk, new longevity derivatives have been proposed. Even if policy makers and academic researchers have been talking about it for many years, securities related to longevity are not largerly traded in markets, especially due to pricing difficulties. In this section, are compared different existing pricing methods. Longevity risk pricing is determined for Survival-forward, but can be used to price other longevity-related instruments. The Hull and White model built in section 2 is used to express the evolution of mortality over time. Are used Italian data to compute the price of the S-forwards according to different three different pricing approaches. Some writers introduced a new method inspired by Solvency II: the use of the cost of capital method to set the price of longevity derivatives. Levantesi and Menzietti [6] proposed a version of this method in a discrete time model. In this section, is described this method, using continuous-time processes to model longevity risks because they provide analytical ease of processing and are generally used in financial instrument pricing. Finally is studied the consistency between the cost of capital method and the three other classical pricing methods often applied in finance,(see 1.4.4): the Wang transform, the Sharpe ratio and the risk-neutral pricing.

3.1 S-forwards

As has been mentionened in the subsection 1.4.4, survival swaps are the most interesting derivative that can be built, because of their low cost of transiction and because they could be customized for the generic buyer, they also may be considered as a collection of simpler derivative: S-forwards. As a results, this section is dedicated on the structure and on the pricing of these derivatives.

3.1.1 Structure

Similar to a classic forward contract, S-forward's two counterparties agree to exchange two payments on a predetermined future date T (maturity date). The cash flow is linked to survival rate index. The buyer of the S-forward pays the seller a fixed rate (the survival rate agreed at the inception of the contract), and gets a variable amount proportional to the realized survival rate in return. Therefore, one counterparty must pay the amount related to the difference between the forward rate and the realized rate at maturity. The forward rate is specified at the beginning of the contract and reflects the expected level of life in the future. If the difference in T between realized survivor and expected survivor is positive, then the insurer suffers a loss in its portfolio. At the beginning of contract , the S-forward value Sf, is zero, with:

$$Sf(T) = [N({}_Tp_x -_T \hat{p}_x)] \tag{87}$$

with N the Notional, ${}_Tp_x$ the floating term, (the realized survival rate in T), and ${}_T\hat{p}_x$ the fixed survival rate. i.e.

$$\mathcal{V}(0, Sf(T)) = 0 \tag{88}$$

With $\mathcal{V}(0, x)$ the value's function at time 0. For a easier progress is considered $N = 1$, and the payoff Sf in T is given by:

$$Sf(T) =_T p_x -_T \hat{p}_x \tag{89}$$

The realized survival rate is adapted to the filtration $\mathcal{F}_T$ (is $\mathcal{F}_T$ measurable), the fixed one is $\mathcal{F}_0$ measurable, since is the probability computed at the beginning of the conctract that an individual aged x in 0 survives T years.

3.1.2 Premise

A S-forward has to be priced. Are needed some hypothesis:

- for the modelling is implemented the Hull and White process (75), choice based on the results in section 2.
- the interest rate term structure implemented is the EIOPA's, based on the Smith and Wilson model with an ultimate forward rate =3.6% [4]
- is indicated with $\mathcal{V}(t, Sf(T))$ the price in t, $\forall$ t in (0,T) of a Survival forward with maturity T.
- named $S(x + t, T - t)$ the survival index in t of an individual aged x, surviving T-t years more: (see Zeddouk and Devolder [43])

$$S(x + t, T - t) = e^{-\int_t^T \mu_x(u)du} \tag{90}$$

[4] see Appendix C

so that under the real world measure $\mathbb{P}$:

$$\mathbb{E}_t^{\mathbb{P}}[S(x+t,T-t)] = e^{A(t,T)-B(t,T)\mu_x(t)} \tag{91}$$

where A(t,T) and B(t,T) solutions of the Riccati equation solved for the Hull and White model in (83)

- are not considered counterparty risk and the basis risk (i.e. the risk related to the fact that the insurer's population differs from the one of the entire population used for the construction of the security).

As noted in the introduction of this section, the price of longevity-linked securities is far from being solved. The literature present on this subject is wide and there is a debate about the quality of the different pricing methods. In fact, there are at least four pricing methods for longevity-linked securities: the Wang transform, the Sharpe ratio approach, the risk-neutral's and the new Cost of Capital method. The first is based on risk-adjusted probablities given by a distortion operator, there are anyway complications in the estimation of the parameter, linked to the lack of an annuity market (see 1.4.4). The second utilizes Sharpe ratio rule, which asserts that the risk premium necessary for investors to tolerate longevity risks equals Sharpe ratio of other undiversifiable financial products (see Milevsky et al. (2005) [44]). This process involves converting the probability measure from a real-world measure to a risk-adjusted one produced by the constant market price of risk (Bauer et al. 2010 [4]). Milevsky (et al. (2006) [45]) recommends a Sharpe ratio of 0.25 for the longevity market based on stock market statistics. However, he stated that "empirical evidence reveals that the risk premium of equities is significantly larger than that of other securities." The Sharpe ratio could also be adjusted with an appropriate annuity quotation, however the difficulty is that the annuity market prices of longevity-related assets are insufficient. In this sense, it should be noted that, in addition to longevity risks, insurance companies are affected by other types of uncertainty, and the resultant risk premium should be larger than the risk premium of the securities linked to the longevity. The third technique is adapting the risk-neutral pricing methodology created for interest rate derivatives to longevity-related variables (see Dahl (2004) [30] ; Cairns et al. (2006) [46], [47]; Biffis et al. (2010) [48]). As Barrieu et al. wrote (Barrieu and Loubergé (2013) [49]) "the main underlying assumption of this approach is that it is possible to replicate cash flows of a given transaction dynamically using basic traded securities in a highly liquid market". As a result, using a risk-neutral method necessitates a highly liquid underlying market, which is required to build a replication strategy. However, present liquidity in the longevity market is extremely low. It is challenging to standardize risk-neutral indicators due to the scarcity of market pricing data for longevity risks. The CoC approach (see Levantesi and Menzietti, [6]) is based on the information included in the RM computed using the conventional Solvency II calculation, can predict the maximum market price of longevity risk. As a result, it enables the adoption of a risk-neutral technique, which is often regarded as the most acceptable way since it is adaptable to many cohorts and ages.

3.1.3 Pricing by Cost of Capital

The Cost of Capital approach concept is based on tying the price of the longevity-linked securities to the capital the insurer must retain to cover unexpected loss according to Solvency II Directive, where is stated that liabilities that cannot be hedged shall be equal to the sum of a best estimate and a risk margin. The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure. The risk margin shall be such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations (see 1.4.2). The method followed by the regulator to calculate the RM is the cost of capital of the basic own founds equal to the SCR, i.e. the capital required to cover the unexpected loss on a one-year time horizon wit 99.5% probability. The CoC rate is the average spread over the risk-free rate that the market expects insurance companies to earn on their equity. The exact computation of RM, required the determination of SCR_t; compute $\forall$ future t the SCR_t would have been too complex for practical purposes, the EIOPA guidelines listed some approximations. Is considered in this context the approximation that assumes that the mortality evolves up to t like its best estimate; so that RM_t is defined as:

$$RM_t = 6\% \sum_{i=t}^{T-1} SCR_i v(t, i+1) \tag{92}$$

The CoC rate is 6% as now is fixed for the SII standard formula, the discount factor is the risk-free. In this context entering a Survival-forward means that the insurer can lower his own exposition (if the cover of the longevity risk is total its SCR is nill and so does the RM from (92)). Thus is evident that at most the price of $Sf(t,T)$ is equal to the cost of retain the SCR, otherwise nobody will be interesed in its buying. So this method leads to the maximum price the insurer is ready to pay for transfer longevity. Based on the considerations above the price corresponds to:

$$Sf^{CoC}(t,T) = BE_t^{\mathbb{P}} + RM_t \tag{93}$$

with

$$BE_t^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{P}}[Sf(T,T)]v(t,T) \tag{94}$$

In analogy, at 0:

$$Sf^{CoC}(0,T) = BE_0^{\mathbb{P}} + RM_0 \tag{95}$$

with

$$BE_0^{\mathbb{P}} = \mathbb{E}_0^{\mathbb{P}}[S(x,T) -_T \hat{p}_x]v(0,T) \tag{96}$$

Instead for the calculation of the RM is used (92) with the same approximation used by Levantesi and Menzietti for determine the succesive $S\hat{C}R_i = SCR_i|_0$:

$$\begin{aligned} SCR_i &= VaR_{99.5\%}[v(i,T)(Sf(T,T)^{99.5\%} - BE_i^{\mathbb{P}})] \\ &= v(i,T)VaR_{99.5\%}[S(x,T) -_T \hat{p}_x] - v(i,T)(\mathbb{E}^{\mathbb{P}}[S(x,T) -_T \hat{p}_x]) \\ &= VaR_{99.5\%}[v(i,T)(S(x,T) -_T \hat{p}_x)] - v(i,T)(\mathbb{E}^{\mathbb{P}}[S(x,T) -_T \hat{p}_x]) \\ &= v(i,T)(S(x,i)(VaR_{99.5\%}[S(x+i,T-i)] - \mathbb{E}^{\mathbb{P}}[S(x,T)]) \end{aligned} \tag{97}$$

$$SCRi|_0 = v(i,T)(\mathbb{E}^{\mathbb{P}}[S(x,i)]VaR_{99.5\%}[S(x+i,T-i) - \mathbb{E}^{\mathbb{P}}[I(x,T)]) \tag{98}$$

Then the price of the S-forward with the Levantesi and Menzietti specification is equal to:

$$\begin{aligned} Sf^{CoC}(0,T) &= v(0,T)(\mathbb{E}^{\mathbb{P}}[S(x,T) -_T \hat{p}_x] \\ &\quad + 6\% \sum_{i=0}^{T-1} v(i,T)v(0,i+1)(\mathbb{E}^{\mathbb{P}}[S(x,i)]VaR_{99.5\%}[S(x+i,T-i)] - \mathbb{E}^{\mathbb{P}}[S(x,T)]) \end{aligned} \tag{99}$$

Applying the Hull and White model defined in premise, to obtain the specific formulation of the price, we must calculate:
$VaR_{99.5\%}[S(x+i,T-i)]$
As seen in section 2, under the measure $\mathbb{P}$:

$$S(x+t,T-t) = e^{-\int_t^T \mu_x(s)ds} = e^{Y(t,T)} \tag{100}$$

Is proved that:

$$Y(t,T) \sim N(\nu(t,T), \eta(t,T)^2) \tag{101}$$

As the survival index S is lognormal and the parameters $\nu(t,T)$,$\eta(t,T)^2$:

$$\begin{cases} \nu(t,T) & = \mu_x(t)\frac{e^{-\alpha(T-t)}-1}{\alpha} - \frac{\kappa e^{\beta t}}{\beta(\alpha+\beta)}(e^{\beta(T-t)}-1) - \frac{\kappa e^{\beta t}}{\alpha(\alpha+\beta)}(e^{-\alpha(T-t)}-1) \\ \eta(t,T)^2 & = \frac{\sigma^2}{\alpha^2}(T-t-\frac{1-e^{-\alpha(T-t)}}{\alpha} - \frac{(1-e^{-\alpha(T-t)})^2}{2\alpha}) \end{cases} \tag{102}$$

So the Value at Risk of the survival index on a confidence level $\varrho = 99.5\%$ is:

$$VaR_{99.5\%}[S(x+t,T-t)] = e^{\nu(t,T)+z_\varrho \eta(t,T)} \tag{103}$$

with z_ϱ the quantile of level ϱ of a standardized normal r.v., so $z_\varrho = 2.58$ (see Solvency II standard formula). So if is required the price in 0 of the S-forward with maturity T, evaluated with the CoC approach with the Levantesi and Menzietti specification in a continuous-time setting, with the application of the Hull and White model:

$$\begin{aligned} Sf^{CoC}(0,T) = {} & v(0,T)(e^{A(0,T)^{HW}-B(0,T)^{HW}\mu_x(0)} - {}_T\hat{p}_x) \\ & + 6\% \sum_{i=0}^{T-1} v(i,T)v(0,i+1)[e^{A(0,i)^{HW}-B(0,i)^{HW}\mu_x(0)}(e^{\nu(i,T)+z_\varrho\eta(i,T)}) \\ & - (e^{A(0,T)^{HW}-B(0,T)^{HW}\mu_x(0)})] \end{aligned} \tag{104}$$

Is important to evaluate the $Sf^{CoC}(0,T)$ not just in 0, but $\forall t$ in (0,T):

$$\begin{aligned} Sf^{CoC}(t,T) = {} & v(t,T)S(x,t)(\mathbb{E}^{\mathbb{P}}[S(x+t,T-t)] \\ & + 6\% \sum_{i=t}^{T-1} v(i,T)v(i,i+1)(\mathbb{E}^{\mathbb{P}}[S(x+t,i-t)]VaR_{99.5\%}[S(x+i,T-i)] - \mathbb{E}^{\mathbb{P}}[S(x,T) \end{aligned} \tag{105}$$

called ${}_tp_x^{obs}$ as the observed survival funciont $\mathcal{F}_t$ measurable equal to $e^{-\int_0^t \mu_x(s)ds}$ from (32), we have for the Hull and White application:

$$
\begin{aligned}
Sf^{CoC}(t,T) = v(t,T)({}_tp_x^{obs} e^{A(t,T)^{HW} - B(t,T)^{HW}\mu_x(t)} - {}_T\,\hat{p}_x) \\
+ 6\% \sum_{i=0}^{T-1} v(i,T)v(0,i+1)[{}_tp_x^{obs} e^{A(t,i)^{HW} - B(t,i)^{HW}\mu_x(t)} (e^{\nu(i,T)+z_\varrho \eta(i,T)}) \\
- (e^{A(t,T)^{HW} - B(t,T)^{HW}\mu_x(t)})]
\end{aligned}
\tag{106}
$$

3.1.4 Pricing by Wang transform

The Wang transform technique is a distortion method based on a distortion operator (1.4.4). In the context of insurance, this operator transforms the best estimate of the survival index into its risk equivalent by applying a certain risk premium. This approach is described as a pricing method utilized in the pricing of numerous over-the-counter longevity swaps in practice (see Dowd et al. [50]).
The distortion risk measure is:

$$\Psi_g(x) = \int_0^\infty g(\overline{F}_x(u))du \tag{107}$$

x continuous (non-negative)random variable $\overline{F}_x(u)$ is its decumulative function and g is the distortion function seen in 1.4.4, given by:

$$g(u) = \Phi[\Phi^{-1}(u) + \pi] \tag{108}$$

Replacing (108) in (107):

$$\Psi_g(S(x+t,T-t)) = \int_0^\infty \Phi[\Phi^{-1}(\overline{F}_{S(x+t,T-t)}(u)) + \pi]du \tag{109}$$

The price in t of a Survival-forward is then:

$$Sf^{Wang}(t,T) = v(t,T)({}_tp_x^{obs}\Psi_g(S(x+t,T-t)) - {}_T\,\hat{p}_x|\mathcal{F}_t) \tag{110}$$

To derive an explicit formulation of the S-forward price using the HW model, we must first calculate the $\Psi_g(S(x+t,T-t))$ formula. With a corrected drift value, the Wang distortion of a log-normal distribution remains log-normal:

$$e^{N(\nu(t,T),\eta^2(t,T))} \longmapsto e^{N(\nu(t,T)+\pi\eta(t,T),\eta^2(t,T))}$$

Thus:

$$\Psi_g(S(x+t,T-t)) = e^{\nu(t,T)+\pi\eta(t,T)+\frac{\eta^2(t,T)}{2}} \tag{111}$$

Then the price of the derivative with Wang under Hull and White is computer as:

$$Sf^{Wang}(t,T) = v(t,T)({}_tp_x^{obs}e^{\nu(t,T)+\pi\eta(t,T)+\frac{\eta^2(t,T)}{2}} -_T \hat{p}_x) \tag{112}$$

3.1.5 Pricing by Sharpe Ratio

The first to apply this pricing approach in this context were Milevsky et al.[44], followed by Loeys et al. [51] for the q-forwards pricing. The Sharpe ratio method is a standard deviation principle that does not employ a change of survival probability. In dynamic financial markets, the Sharpe ratio serves as a baseline for determining risk premium. The Sharpe ratio is provided by:

$$\pi_{Sharpe} = \frac{\mathbb{E}[R_p - R_f]}{\sigma_p} \tag{113}$$

$\mathbb{E}[R_p - R_f]$ is the expected spread between the portfolio return and the risk-free rate. σ_p is the standard deviation of the portfolio return. In the insurance environment (see Barrieu et al., 2014 [52]) (113) can be expressed as:

$$\pi_{Sharpe} = \frac{\mathcal{V}(T,Y) - \mathbb{E}[Y]}{\sigma(Y)} \tag{114}$$

Where $\mathcal{V}(T,Y)$ is the value in T of the longevity-linked Y (its payoff), $\mathbb{E}[Y]$ its Best Estimate, and $\sigma(Y)$ its standard deviation. Under this specification, the price of $Sf(t,T)$ is:

$$\begin{aligned} Sf^{Sharpe}(t,T) = v(t,T)({}_tp_x^{obs}\mathbb{E}^{\mathbb{P}}[S(x+t,T-t)] -_T \hat{p}_x \\ + \pi_t p_x^{obs}\sqrt{\mathrm{Var}^{\mathbb{P}}[S(x+t,T-t)]}) \end{aligned} \tag{115}$$

with $\mathrm{Var}^{\mathbb{P}}[S(x+t,T-t)]$ the variance of the survival index, π the supposed sharp ratio. Considering the Hull and White model, the variance of the survival function could be written as:

$$\begin{aligned} \mathrm{Var}^{\mathbb{P}}[S(x+t,T-t)] = \mathrm{Var}^{\mathbb{P}}(e^{Y(t,T)}) = \mathrm{Var}^{\mathbb{P}}(e^{N(\nu(t,T),\eta(t,T)^2)}) \\ = (e^{\eta(t,T)^2} - 1)e^{2\nu(t,T)+\eta(t,T)^2} \end{aligned} \tag{116}$$

$$Sf^{Sharpe}(t,T) = v(t,T)({}_tp_x^{obs}\mathbb{E}^{\mathbb{P}}[S(x+t,T-t)] -_T \hat{p}_x + \pi_t p_x^{obs}\sqrt{(e^{\eta(t,T)^2}-1)e^{2\nu(t,T)+\eta(t,T)^2}}) \tag{117}$$

3.1.6 Pricing by Risk Neutral

Risk-Neutral pricing is a method extensively used in quantitative finance to generate derivative prices, it may be applied to compute the price of longevity-linked securities. As widely noted in several studies this methodology is not appropriate for pricing longevity derivatives since the longevity market is still in its infancy, and this method necessitates the availability of a significant amount of data.

Risk Neutral measure - Girsanov Theorem
Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a non negative r.v. Z $\diagup$ $\mathbb{E}[Z]$=1 . Be defined a new probability measure $\mathbb{Q}$ where:

$$\mathbb{Q}(A) = \int_A Z(\omega)dP(\omega) \tag{118}$$

$\forall A \in \mathcal{F}$. Any r.v. X has now two expectations, one under the original measure $\mathbb{P}$ and one under $\mathbb{Q}$. The relations between the two measures is given by

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[XZ] \tag{119}$$

If $\mathbb{P}(Z > 0) = 1$ and so does under $\mathbb{Q}$ (119) could be written as:

$$\mathbb{E}^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{Q}}[\frac{X}{Z}] \tag{120}$$

Z is the Radon-Nikodym derivative of $\mathbb{Q}$ respect to $\mathbb{P}$, and:

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{121}$$

Lemma. Be t in [0,T] and let Y be a $\mathcal{F}_t - measurable$ r.v.. Then

$$\mathbb{E}^{\mathbb{Q}}[Y] = \mathbb{E}^{\mathbb{P}}[YZ(t)] \tag{122}$$

Lemma. Be t and s in $0 \leqslant s \leqslant t \leqslant T$ and let Y be a $\mathcal{F}_t - measurable$ r.v.. Then

$$\mathbb{E}^{\mathbb{Q}}[Y|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}^{\mathbb{P}}[YZ(t)|\mathcal{F}_s] \tag{123}$$

Girsanov Theorem - one dimension. Be Z_t, for $0 \leqslant t \leqslant T$ a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t$ a filtration for the brownian. Let B_t for $0 \leqslant t \leqslant T$ an adapted process. Define:

$$\rho_t = exp(-\int_0^t B_u dZ_u - \frac{1}{2}\int_0^t B_u^2 du) \tag{124}$$

$$\tilde{Z}_t = Z_t + \int_0^t B_u du \tag{125}$$

Is also assumed that

$$\mathbb{E}[\int_0^T B_u^2 \rho_u^2 du] < \infty \tag{126}$$

Set $\rho = \rho(T)$. Then $\mathbb{E}[\rho]$=1 and under the probability measure $\tilde{\mathbb{P}} = \mathbb{Q}$ given by (118), $\tilde{Z}_t$ is a Brownian motion.
Is setted up an asset price model in which $\mathbb{P}$ is the actual probability measure and $\mathbb{Q}$ is the risk-neutral measure. Considering a payoff Y_T not exposed to interest rate risk. If the market is perfect complete and without arbitrages (the Solvency II requirements), then the value in 0 of Y_T is:

$$\mathcal{V}(0, Y_T) = \frac{\mathbb{E}_0^{\mathbb{Q}}[Y_T]}{[1 + i(0,T)]^T} \tag{127}$$

The risk neutral expectation is the market equivalent, i.e. the RN probability measure contains the risk premium and so is a risk-adjusted measure. Based on what stated before in Girsanov theorem, $\mathbb{Q}$ is also known as equivalent martingale measure, specified in corrispondence of a numerary $\mathcal{N}$, that is the unit of measurement of the measure, $\exists$ a one-to-one corrispondence between the couple (Measure, Numerary), for the measure $\mathbb{Q}$ the numerary is the money market account δ:

$$\delta_T = e^{\int_0^T r(u)du} \tag{128}$$

r(u) spot rate. So the money market account is the payoff in T of an investment in t=0 of a one unit of currency in a roll-over strategy in ZCB with infinitesimal maturity. If the market is complete perfect and free from arbitrages than $(\mathbb{Q}, \delta)\exists!$. Because the longevity-linked securities market has not these characteristic the risk neutral approach proves difficulties. Is useful define a $\mathbb{Q}_\pi$ probability measure, where π s the market price of longevity risk. Respect what stated the price in t of an S-forward under the risk-neutral approach can be written as:

$$Sf(t,T)^{\mathbb{Q}_\pi} = v(t,T)({}_tp_x^{obs}\mathbb{E}_t^{\mathbb{Q}_\pi}[S(x+t, T-t)] -_T \hat{p}_x) \tag{129}$$

To reach the explicit formula of (129) using the Hull and White model for the modelling of $\mu_x(t)$ in a continuous-time setting, is needed the introduction of a function of time of the market prices, said $\pi(t, \mu_x(t))$, and let $\tilde{Z}_t$ defined as (125) with the B_t function replace by $\pi(t, \mu_x(t))$, following the Zeddouk's proceeding [43] is put $\pi(t, \mu_x(t)) = \pi$ a constant market price of risk. Under the probability measure $\mathbb{Q}_\pi$ the Hull and White process manteins his affines proprierties and can be written as:

$$d\mu_x^{\mathbb{Q}_\pi}(t) = (\vartheta - \alpha\mu_x(t) + \sigma\pi)dt + \sigma\tilde{Z}_t \tag{130}$$

Where as stated in the Girsanov Theorem $\tilde{Z}_t$ is a standard Brownian motion under $\mathbb{Q}_\pi$ probability measure. From (130) a positive value of π increases the intensity of mortality. Under risk neutral probability the survival probability is given by:

$$\mathbb{E}^{\mathbb{Q}_\pi}[S(x+t, T-t)] = e^{A^{HW}_{\mathbb{Q}_\pi}(t,T) - B^{HW}_{\mathbb{Q}_\pi}(t,T)\mu_x(t)} \tag{131}$$

whit $A^{HW}_{\mathbb{Q}_\pi}(t,T)$ and $B^{HW}_{\mathbb{Q}_\pi}(t,T)$ given by

$$\begin{cases} A^{HW}_{\mathbb{Q}_\pi}(t,T) & = \frac{\kappa}{\alpha}[e^{-\alpha T}\frac{e^{(\beta+\alpha)T} - e^{(\beta+\alpha)t}}{\beta+\alpha} - \frac{e^{\beta T} - e^{\beta t}}{\beta}] - \frac{\sigma^2}{2\alpha^2}[\frac{1}{\alpha}(1 - e^{-\alpha(T-t)}) - T + t] \\ & -\frac{\sigma^2}{4\alpha^3}(1 - e^{-\alpha(T-t)})^2 - \frac{\sigma\pi}{\alpha}(1 - exp(-\alpha(T-t)) \\ B^{HW}_{\mathbb{Q}_\pi}(t,T) & = \frac{1}{\alpha}(1 - e^{-\alpha(T-t)}) \end{cases} \tag{132}$$

So the price with explicit formulation of the Hull and White model:

$$Sf(t,T)^{\mathbb{Q}_\pi} = v(t,T)({}_tp_x^{obs} e^{A^{HW}_{\mathbb{Q}_\pi}(t,T) - B^{HW}_{\mathbb{Q}_\pi}(t,T)\mu_x(t)} -_T \hat{p}_x) \tag{133}$$

3.1.7 Numerical Application

Are now priced S-forwards with the three differents approaches studied for Italian population. Are stated the following hypotesis:

- Given the italian cohort of individuals born in 1955 that had 65 years old in 2020
- N_0=10 000 policyholders
- Payment of 1£ to each policyholder alive in T
- It is priced the S-forward issued in $t = 2020 = 0$ for T= 5,10,15

- The fixed leg is based on IPS55 built by ANIA
- The risk free structure is given by EIOPA's in 2020

The following table contains the fixed survival rates from IPS55, $\mu_{65}(0) = 0.008861649$

T	**5**	**10**	**15**
${}_T\hat{p}_{65}$	0.9488647	0.8775302	0.774194

Table 8: Fixed survival rates ${}_T\hat{p}_x$ for T=5,10,15 for cohort of 1955 in 2020

$\mu_{65}(0)$	α	κ	β	σ
0.008861649	0.1721555423	0.001525582	0.10983905356	0.0018082860

Table 9: Optimal parameters for the survival process Hull and White model

Hull and White			
$N_0 = 10000$	$x_0 = 65$	$\mu_{65}(0) = 0.008861649$	
T	**Price**	**BE %**	**RM%**
5	16.082326	83.68914%	16.31086%
10	29.77604	63.87287%	36.12713%
15	47.79294	59.30265%	40.69735%

Table 10: *Prices of different S-forwards via CoC, maturity= 5,10,15, technical rate=0*
Number of individuals whose longevity to be hedged $= N_0 = 10000$

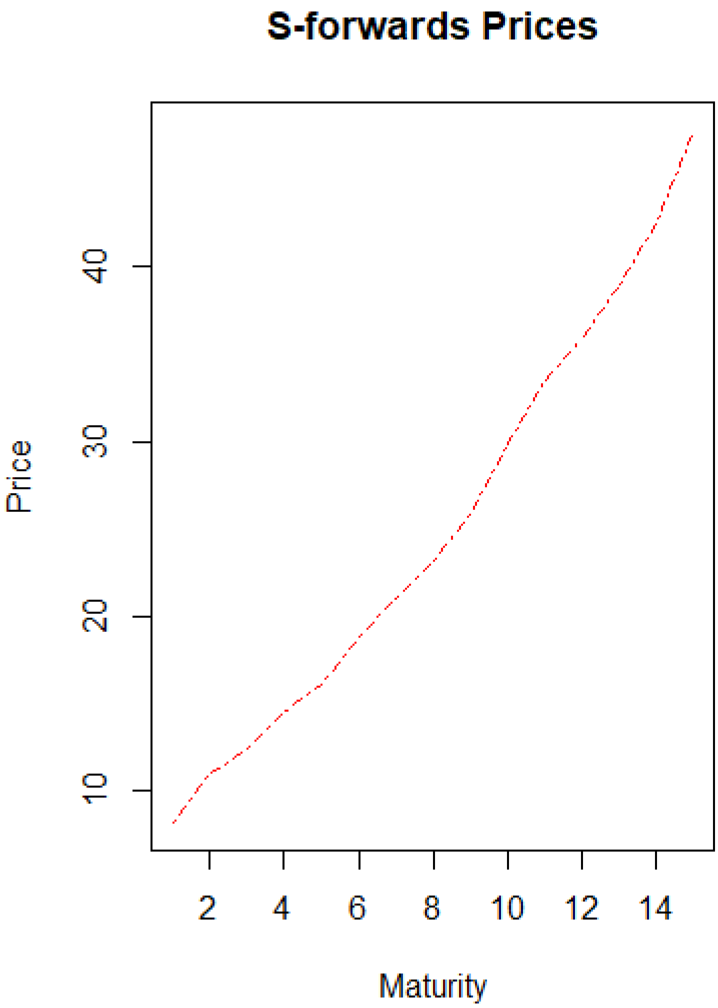


Figure 22: *Prices S-forwards, T=1,...,15, 1955 cohort via Cost of Capital*

Comparison with other pricing methods

To test the consistency of the Levantesi and Menzietti method with the classical pricing methods of (1.4.4), must be computed the parameters that identify these methods in such a way that the Cost of Capital price equals the price of the others techinques. If the parameter is stable then the methods are consistent:

Wang Transform

From 3.1.4 (112), is computed the π of the Wang method that equals the CoC prices for T=5,10,15:

T	CoC Price	π
5	16.082326	9.51%
10	29.77604	9.46%
15	47.79294	10.79%

Table 11: *Wang's parameters π that equals (112) and (106)*

Sharpe ratio

As seen in 3.1.5 (117) the price of an S-forward in 0 with this method, applying the Hull and White model of the section 2, can be computed as:

$$Sf^{Sharpe}(0,T) = v(0,T)(e^{A(0,T)^{HW}-B(0,T)^{HW}\mu_x(0)} - {}_T\hat{p}_x + \pi\sqrt{(e^{\eta(0,T)^2}-1)e^{2\nu(0,T)+\eta(0,T)^2}}) \tag{134}$$

Therefore, is possible to determine π for T=5,10,15:

T	CoC Price	π
5	16.082326	9.53%
10	29.77604	9.47%
15	47.79294	11.89%

Table 12: *Sharpe's parameters π that equals (117) and (106)*

Risk Neutral

For this application is needed the market price of longevity risk, it has been cmputed as the market price that equals (106). As seen in 3.1.6 (133) the price of an S-forward in 0 with this method, applying the Hull and White model of the section 2, can be computed as:

$$Sf(0,T)^{\mathbb{Q}_\pi} = v(0,T)(e^{A^{HW}_{\mathbb{Q}_\pi}(0,T)-B^{HW}_{\mathbb{Q}_\pi}(0,T)\mu_x(0)} -_T \hat{p}_x) \qquad (135)$$

T	CoC Price	π
5	16.082326	-14.54%
10	29.77604	-21.47%
20	47.79294	-29.89%

Table 13: *Equivalent market price π that equals (133) and (106)*

3.2 Conclusion

In this article, a new technique to price longevity linked securities based on the Cost of Capital approach has been proposed in a continuous-time setting, in accordance with the Solvency II requirement. Has been cited as an example of a stochastic time-continuous process for lifespan the Hull and White model. The Cost of Capital method is defined by a single variable: the Cost of Capital rate, which is imposed by EIOPA (6 percent in Solvency II now), therefore is not needed its estimation. This single parameter value may be used as a benchmark, allowing us to identify similar parameter values in the three other techniques. The Levantesi and Menzietti method estimates the "maximum price" that the fixed payer is willing to pay for hedging longevity risk. The annuity provider would have the convenience of retaining longevity risk if the necessary S-forward price were greater. The simulations presented above show that the Wang and the Sharpe ratio approaches are more consistent than the Risk Neutral with the CoC (see the following Figure 19).
It can be noted that the Cost of Capital is one way to calculate the Risk Margin, but this entity could be measured in other ways (e.g. with a percentile method at 75% of the BE), indeed when in 1.4.2 are listed the Solvency II articles, is the regulator that suggest the CoC apprach to determine the RM, but being a prudential amount over the BE, it could have been determined in different ways. In addition it has been assumed that all insurance firms accept the same market price for longevity risk, which is included in the Solvency II RM calculation. However, diversification effects, strategic considerations, and risk attitude may drive the insurer to accept a different market price for longevity risk. Furthermore, the counterparty default risk has not been

included, which would raise the price of longevity-linked instruments and need a separate capital charge for the insurer. Further study should be conducted to determine the impact of this risk on the risk margin and the maximum price of longevity risk. Even though this technique has certain drawbacks, it has the benefit of operating inside the well-known and potentially standardized framework of Solvency II, and the RM can be regarded a potential instrument for estimating a maximum value for the market price of longevity risk.

T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Prices	8.234569	10.9385	12.38492	14.49474	16.08233	18.79201	20.98391	23.20003	25.8593	29.77604	33.34738	35.94852	38.99302	42.39023	47.79294
CoC/Wang	15.35%	11.78%	10.55%	9.98%	9.51%	9.50%	9.48%	9.42%	9.44%	9.46%	9.68%	9.83%	9.98%	10.04%	10.79%
CoC/Sharpe	15.36%	12.88%	10.54%	9.98%	9.53%	9.51%	9.34%	9.41%	9.44%	9.47%	9.68%	9.81%	9.96%	10.68%	11.89%
CoC/RN	-9.14%	-10.33%	-12.20%	-13.40%	-14.54%	-16.93%	-17.32%	-19.22%	-19.78%	-21.47%	-22.61%	-24.43%	-28.66%	-29.54%	-29.89%

Figure 23: *S-forward prices under the Cost of Capital approach, Consistency between Wang/Sharpe/Risk-Neutral and Cost of Capital method HW model*

A Brownian Motion and Stochastic Calculus

Given $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}_t$ on it, is called standard Brownian Motion $Z = (Z_t, t \geqslant 0)$ or Z(t) the stochastic process that has the following properties:

- $P(Z_0 = 0) = 1$
- $Z_t - Z_s$ is indipendent from $\mathcal{F}_s$:
  for t_k in $0 = t_0 < t_1 < ... < t_k < ... < t_n = t \longrightarrow Z(t_k) - Z(t_{k-1})$ indip. rv. The process has indipendent increments
- The r.v. $Z_t - Z_s$ has normal distribution with mean 0 and variance $|t - s|$, i.e.
  $Z_t - Z_s \sim N(0, |t - s|)$.

A process with indipendent increments is always Markovian:
$t > s$; $P(Z_t \in A|\mathcal{F}_s)$=$P(Z_t \in A|Z_s)$ $\Leftrightarrow$ the process depends only on the position in s. If Z_s in known, Z_z and Z_t with $z < s < t$ are conditionally indipendent.

The Brownian motion is a Martingale:

$$\mathbb{E}[Z_t|\mathcal{F}_s] = \mathbb{E}[Z_t - Z_s + Z_s|\mathcal{F}_s] = \mathbb{E}[Z_t - Z_s] + Z_s = Z_s \tag{136}$$

The Markovian and the Martingale properties comes from the indipendent increases'property

Tansitional law:

$$P(Z_t \in dy|Z_s = x) = \frac{e^{\frac{-(y-x)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}dy \tag{137}$$

The trajectories of the Brownian motion are characterized:

- continuous
- not differentiable
- quadratic deviation of the order of a time

The first item could be proof from a Kolmogorov theorem:
Kolmogorov Theorem Given the process X(t), $t \geqslant 0$ and be given $g = g(h)$ and $q = q(h)$ two non-decreasing monotone even functions for $h > 0$, and $\sum_{n=1}^{\infty} g(2^{-n}) < \infty$, $\sum_{n=1}^{\infty} 2^n q(2^{-n}) < \infty$;
if $\forall t, t+h \in (a,b)$, $P(|X(t+h) - X(t)| > g(h)) < q(h)$,
then X(t) ha continuous trajectories.

Continuous and non-differentiable curves were articulated by mathematicians already in the last century (e.g. the Von Knocke curve) and were presented as pathological phenomena. Brownian motion has trajectories that are all (with the exception of a set of zero probabilities) similar to those by Von Knocke. A continuous undifferentiable trajectory is a fractal (non-integer dimension).
A famous result due to Lévy shows that the quadratic variation of the Brownian motion in the time interval [0, t] is equal to t. This result is related to the fact that the variation before motion Brownian in [0, t] is infinite. Obviously this latter property derives from the non-differentiability of its trajectories:
Theorem For the Brownian Z_t:

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} [Z(kt2^{-n}) - Z((k-1)t2^{-n})]^2 = t \tag{138}$$

Stochastic integration and Itô's formula

The integration of stochastic processes respect to Brownian motion was introduced by several authors in a completely independent way. The credit is attributed to Kiyosi Itô, but it seems that before him the idea had also come to Wolfgang Doeblin and Iosif Gikhman. The study of stochastic differential equations was born already in 1940s and had a major impact on theory and applications both to physical sciences and to finance and economics.

Given $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $\mathcal{F}_t$ on it and a standard Brownian Motion $Z = (Z_t, t \geqslant 0)$ on the filtration. Given X such that:

$$X : ([0,T] \times \Omega, \mathcal{B}[0,T] \times \mathcal{F}_\infty) \longrightarrow (\mathbb{R}, \mathcal{B}) \tag{139}$$

This means that $\forall (t,\omega) \in [0,T]$ is defined the application $(t,\omega) \longrightarrow X(t,\omega)$. As $t \in [0,T]$ varies, it is described the trajectory ω while, for fixed t, $X(t,\omega)$

represents the set of values (or positions) of process X at time t. The class of processes for which we define the stochastic integral respect to Z in the interval [0, T] is indicated with the symbol $\mathcal{H}_2[0,T]$ and has the following properties:
(i) for every $t \in [0,T], X(t,\omega)$ is $\mathcal{F}_t$-measurable, that is, for every Borel's set $A \in \mathcal{B}[0,T]$, we have that $(\omega : X(t,\omega) \in A) \in \mathcal{F}_t$;
(ii)

$$\int_0^T \mathbb{E}[X^2](t)dt = \int_0^T dt \int_\Omega X^2(t,\omega)dP(\omega) < \infty \tag{140}$$

Property (i) indicates that processes X must be non-anticipating, that is, their values must depend only on the events that have occurred until instant t. Property (ii) concerns the finiteness of the second moment of the process X referred to by the notation $\mathcal{H}_2[0,T]$. The construction of the stochastic integral follows the path of constructionclassical of the Lebesgue-Stieltjies integral. The first step is in constructing the stochastic integral for simple processes, that is processes having the following form:

$$\begin{aligned} X(t,\omega) &= \sum_{i=0}^{n-1} X(t_i,\omega)1_{(t_i,t_{i+1}]}(\omega) \\ &= \sum_{i=0}^{n-1} a_i 1_{I_i}(\omega) \end{aligned} \tag{141}$$

where 1_A is the indicator function of the set A and $I_i = (t_i, t_{i+1}], \bigcup_{i=0}^{n-1} I_i = (0,T]$ and $I_i \cap I_j = \emptyset$ for $i \neq j$ and $a_i : \Omega \longrightarrow \mathcal{R}$ are $\mathcal{F}_t$-measurable r.v. with $\mathbb{E}a_i^2 < \infty$.

The simple stochastic integral of X respect to Brownian motion is:

$$\int_0^T X(t,\omega)dZ(t,\omega) = \sum_{i=0}^{n-1} a_i[Z(t_{i+1},\omega) - Z(t_i,\omega)] = \sum_{i=0}^{n-1} a_i \Delta Z_i \tag{142}$$

where $0 = t_0 < t_1 < ... < t_n = T$ The next step in the integral construction is the approximation $\forall X \in \mathcal{H}_2[0,T]$, with simple processes X_n such that $X_n \longrightarrow X$ in $\mathcal{H}_2$:

$$\int_0^T XdZ = \lim_{n\to\infty} q.m. \int_0^T X_n dZ \tag{143}$$

where $\lim_{n\to\infty} q.m.$ is the limit in quadratic mean of X_n.

The propierties of the stochastic integral of a simple precess X:

Theorem

(i) for the process of (142) we have $\mathbb{E}[\int_0^T XdZ] = 0$

(ii) the process $\int_0^T XdZ$, T>0 is a martingale adapted to the filtration $\mathcal{F}_T$, T>0

(iii) $\mathbb{E}[\int_0^T XdZ]^2 = \int_0^T \mathbb{E}[X^2]dt$

(iv) $\mathbb{E}[\sup_{0\leqslant t\leqslant T}(\int_0^t XdZ)^2] = 4\int_0^T \mathbb{E}[X^2]dt$

Itô's Formula

A fondamental result of the stochastic calculus is the Itô's formula. Considering a function $y = f(x,t)$ with $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$, with partial derivatives continuous and limitates. Then for a process $Y(t) = f(Z(t),t)$ the Itô's formula gives a rapresentation of dY as follows:

$$dY = \frac{\partial f}{\partial x}(Z(t),t)dZ + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(Z(t),t)dt + \frac{\partial f}{\partial t}(Z(t),t)dt \tag{144}$$

(144) differs to normal calculus for the term with the second order derivative. The associate dt term discends from the fact that $\Delta^2 Z \simeq \Delta t$.

Supposed that $\frac{\partial f}{\partial x}(Z(s),s) \in \mathcal{H}_2[0,t]$ for s in (0,t). Thus:

$$\begin{aligned} & f(Z(t),t) - f(Z(0),0) \\ & = Y(t) - Y(0) \\ & = \int_0^t \frac{\partial f}{\partial x}(Z(s),s)dZ(s) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(Z(s),s)ds + \int_0^t \frac{\partial f}{\partial t}(Z(s),s)ds \end{aligned} \tag{145}$$

B Affine Processes

For the modellig of the intensity of mortality $\mu()$ has been used in this work affine models:

$$\begin{aligned} d\mu(t) =& f(t,\mu(t))dt + g(t,\mu(t))dZ_t\mathbb{P} \\ d\mu(t) =& \hat{f}(t,\mu(t))dt + g(t,\mu(t))dZ_t\mathbb{Q} \end{aligned} \tag{146}$$

respectively the dynamics of $\mu(t)$ under the real world probability and the risk adjusted. Has been chosen the affine's class model (i.e. f and g^2 are affine in μ) because in this way is possible to solve analytically the stochastic differential equation (under real world and under RN) with the particularly boundary constraint on $S(\mu(t),t)$:

$$\begin{cases} f(t,\mu(t)) = & f_0 + f_1\mu(t) \\ g^2(t,\mu(t)) = & g_0 + g_1\mu(t) \end{cases} \tag{147}$$

The distribution probability of $\mu(t)$ turns stationary with the appropriate choice of the sign of f_0 and f_1. If one reasonably chooses positive f_0 and negative f_1, a mean reverting process is created which opposes the Brownian gdZ which tends to distort the distribution. The two conflicting hypotheses make the process stationary.
For the choice of $g^2(t,\mu(t))$ there are three hypotesis that corresponds to the three classical financial models studied:

$$g^2(t,\mu(t)) = \begin{cases} g_1 = 0 & \Rightarrow g^2(t,\mu(t)) = g_0 \\ g_1 > 0, g_0 = 0 & \Rightarrow g^2(t,\mu(t)) = g_1\mu(t) \\ g_1 > 0, g_0 \neq 0 & \Rightarrow g^2(t,\mu(t)) = g_0 + g_1\mu(t) \end{cases} \tag{148}$$

Respectively Vasicek, CIR (Cox Ingersoll Ross) and traslated CIR.

Solving the SDE with boundary condition ${}_{T-T}p_x = 1$, and with $\hat{f}(t,\mu(t)) = \alpha(\gamma - \mu(t))$ and $g^2(t,\mu(t)) = g_0 + g_1\mu(t)$, we obtain:

$$p(t,T) = e^{-A(\tau)-B(\tau)\mu(t)} \tag{149}$$

with $p(t,T) =_T p_{x+t}$.
This solution is the same for all the affines models, and is proved by substituting it in the SDE:

$$\frac{\partial p(t,T)}{\partial t} + [\alpha(\gamma - \mu(t))]\frac{\partial p(t,T)}{\partial \mu(t)} + \frac{1}{2}g^2(t,\mu(t))\frac{\partial^2 p(t,T)}{\partial \mu(t)^2} = p(t,T)\mu(t) \tag{150}$$

Substituing (149) in (150) are computed the derivatives:

$$\begin{aligned} \frac{\partial p(t,T)}{\partial t} =& e^{-A(\tau)-B(\tau)\mu(t)}\frac{\partial}{\partial t}[-A(\tau)-B(\tau)\mu(t)] = \\ =& p(t,T)[\frac{\partial}{\partial\tau}A(\tau)+\frac{\partial}{\partial\tau}B(\tau)\mu(t)] \end{aligned} \tag{151}$$

$$\frac{\partial p(t,T)}{\partial\mu(t)} = p(t,T)[-B(\tau)] \tag{152}$$

$$\frac{\partial^2 p(t,T)}{\partial\mu(t)^2} = p(t,T)B^2(\tau) \tag{153}$$

i.e.

$$p(t,T)[\frac{\partial}{\partial\tau}A(\tau)+\frac{\partial}{\partial\tau}B(\tau)\mu(t)]+\hat{\alpha}(\hat{\gamma}-\mu(t))p(t,T)[-B(\tau)]+\frac{1}{2}(g_0+g_1\mu(t))p(t,T)B^2(\tau) = p(t,T)\mu(t) \tag{154}$$

thus

$$[\frac{\partial B(\tau)}{\partial\tau}+\hat{\alpha}B(\tau)+\frac{1}{2}g_1B^2(\tau)-1]\mu(t)+[\frac{\partial A(\tau)}{\partial\tau}-\hat{\alpha}\hat{\gamma}B(\tau)+\frac{1}{2}g_0B^2(\tau)] = 0 \tag{155}$$

Both terms in [.] must be 0 so that $p(t,T)$ is solution of the SDE $\forall\mu(t)$:
The first is a differential equation in $B(\tau)$:

$$\frac{\partial B(\tau)}{\partial\tau} = -\frac{1}{2}g_1B^2(\tau)-\hat{\alpha}B(\tau)+1 \tag{156}$$

To be solved is needed a boundary condition for B, from (149) and p(T,T)=1 $\forall\mu$ then:

$$\begin{cases} A(0) = & 0 \\ B(0) = & 0 \end{cases} \tag{157}$$

thus are obtained the known Riccati equations:

$$\begin{cases} \frac{\partial B(\tau)}{\partial\tau} = & -\frac{1}{2}g_1B^2(\tau)-\hat{\alpha}B(\tau)+1 \\ B(0) = & 0 \end{cases} \tag{158}$$

In Vasicek is elementary as $g_1 = 0$

$$\begin{cases} \frac{\partial A(\tau)}{\partial \tau} = & \hat{\alpha}\hat{\gamma}B(\tau) - \frac{1}{2}g_0 B^2(\tau) \\ A(0) = & 0 \end{cases} \tag{159}$$

Once solved (158), (159) is easy solved by integration:

$$\int_0^\tau A(u)du = \hat{\alpha}\hat{\gamma}\int_0^\tau B(u)du - \frac{1}{2}g_0 \int_0^\tau B^2(u)du \tag{160}$$

All these integrals are solvable in closed form, thus $A(\tau)$ and $B(\tau)$ are explicitly solvable

C EIOPA's structure

EIOPA's term structure is not constructed starting from the resolution of an optimal problem between the model and the market, but is based on an interpolation method, setting a limit value (3.6%) to infinity exogenously. EIOPA works directly on discount factors $v(t, t+\tau)$ and looks for a function that passes through all points.

Interpolation

We are looking for a function that passes through n points:

- $n = 2 \longrightarrow$ passes a straight line (n-1 order polynomial: $y = ax + b$)
- $n = 2 \longrightarrow$ passes a parabola (n-1 order polynomial: $y = ax^2 + bx + c$)
- if n is "big" are used Lagrange polynomial:
  if $x = x_k$, $f(x_k) = 1$, if $x = x_i$ with $i \neq k$ $f(x_i) = 0$, e.g.:
  for n=2:

$$
\begin{aligned}
L_1(x) = & \frac{x - x_2}{x_1 - x_2} \longmapsto \begin{cases} if \quad x = x_1 : L_1(x_1) = 1 \\ if \quad x = x_2 : L_1(x_2) = 0 \end{cases} \\
& or \\
L_2(x) = & \frac{x - x_1}{x_2 - x_1} \longmapsto \begin{cases} if \quad x = x_1 : L_2(x_1) = 0 \\ if \quad x = x_2 : L_2(x_2) = 1 \end{cases}
\end{aligned}
\tag{161}
$$

  The line that passes through two points is
  $f(x) = y_1 L_1(x) + y_2 L_2(x)$

  for n=3:

$$
\begin{aligned}
L_1(x) = & \frac{x-x_2}{x_1-x_2}\frac{x-x_3}{x_1-x_3} \longmapsto \begin{cases} if & x=x_1 : L_1(x_1)=1 \\ if & x=x_2 : L_1(x_2)=0 \\ if & x=x_3 : L_1(x_3)=0 \end{cases} \\
& or \\
L_2(x) = & \frac{x-x_1}{x_2-x_1}\frac{x-x_3}{x_2-x_3} \longmapsto \begin{cases} if & x=x_1 : L_2(x_1)=0 \\ if & x=x_2 : L_2(x_2)=1 \\ if & x=x_3 : L_2(x_3)=0 \end{cases} \\
& or \\
L_3(x) = & ...
\end{aligned} \tag{162}
$$

The parabola that passes through three points is
$f(x) = y_1L_1(x) + y_2L_2(x) + y_3L_3(x)$

Generally $f(x) = \sum_{i=0}^{n} y_iL_{n,i}(x)$

Alternatively, one can interpolate piecewise, minimizing the problems related to the representation of the polynomial which could tend to oscillate a lot. The trade-off is in the choice of a third degree polynomial. With the cubic spline, is needed a point of continuity between the sections. So if one has n+1 point $\Rightarrow$ n intervals.
In the i-interval one has $f_i(x) = a_ix^3 + b_ix^2 + c_ix + d_i$.
So 4 parameters for n uknowns (n intervals) $\Rightarrow$ 4n uknows, to which correspond n + 1 conditions, the passage through the n + 1 points minus the two extremes that have no contacts= n+1-2= $n-1$, to which are added 3 (n-1) conditions given by the continuity of the functions, continuity of the first derivative and continuity of the second derivative.

$3(n-1)+n+1 = 4n-2$ conditions, so are needed two more conditions: $f''(x_0) = f''(x_n)$.
The one described is the Natural Cubic Spline. The spline used by EIOPA is the Tension Spline, where there is an exogenous parameter,

are intermediate splines between linears and cubics. The tension splines are characterized by an exogenous parameter, a tension parameter such that if it tends to 0, obtaining the cubic spline, if it tends to infinity, the linear spline is obtained.
EIOPA's Smith and Wilson model interpolate on $v(t, t+\tau)$ and is given by:

$$v(t, t+\tau) = e^{-h_\infty \tau} + \sum_{j=1}^{N} \xi_j W(\tau, \tau_j) \tag{163}$$

for a given τ_j and s, the $W(\tau, \tau_j)$ is such that:

$$W(\tau, s) = e^{-h_\infty(\tau+s)}[\alpha min(\tau, s) - e^{-\alpha max(\tau,s)} sinh(\alpha min(\tau, s)] \tag{164}$$

the parameters are $-h_\infty$, α exogenous, and ξ_i, τ_i given by the market for i=1,...,N
h_∞ is such that $h_\infty = \lim_{\tau \to \infty} h^{SW}(t, t+\tau)$
α is a free-parameter (semi-exogenous), adjusts how quickly the curve reaches the asymptotic value as τ increases.
Given $v(t, t+\tau)^{SW}$ is possible to write $h(t, t+\tau)^{SW} = -\frac{1}{\tau} lnv(t, t+\tau)$, thus:

$$\lim_{\tau \to \infty} -\frac{1}{\tau} lnv(t, t+\tau) = h_\infty \tag{165}$$

To explicitly choose α, it must be made such that
$\tau^* / |h(t, t+\tau^*) - h_\infty| < \varepsilon$

EIOPA does not want the asymptote to be reached in a very high number of years, therefore it imposes an α such that the structure arrives at the asymptote in generally a $\tau^* = 60$.
The other parameters are given from market by:

$$\begin{bmatrix} v(t, t+\tau_1) \\ v(t, t+\tau_2) \\ ... \\ v(t, t+\tau_N) \end{bmatrix} = \begin{bmatrix} e^{-h_\infty \tau_1} \\ e^{-h_\infty \tau_2} \\ ... \\ e^{-h_\infty \tau_N} \end{bmatrix} + \begin{bmatrix} W(\tau_1, \tau_1) + ... + W(\tau_1, \tau_N) \\ W(\tau_2, \tau_1) + ... + W(\tau_2, \tau_N) \\ ... \\ W(\tau_N, \tau_1) + ... + W(\tau_N, \tau_N) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ ... \\ \xi_N \end{bmatrix} \tag{166}$$

or simply:

$$\begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} \eta \end{bmatrix} + \begin{bmatrix} W \end{bmatrix} \begin{bmatrix} \xi \end{bmatrix} \tag{167}$$

i.e:

$$\vec{\xi} = W^{-1}(\vec{v} - \vec{\eta}) \tag{168}$$

Observation: the sinh function
$y(x)$ is the spline function on the whole intervale $[x_0, x_N]$, $\forall$ point, the second order derivative has an important property: it can be written as linear combination of the $y^{''}(x)$ itselves in $x_0, x_1, ...$, i.e in (x_i, x_{i+1}):

$$y^{''}(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} y^{''}(x_i) + \frac{x - x_i}{x_{i+1} - x_i} y^{''}(x_{i+1}) \tag{169}$$

$\exists$ curves such that $[y^{''}(x) - \sigma^2 y(x)]$:

$$y^{''}(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i}[y^{''}(x_i) - \sigma^2 y(x_i)] + \frac{x - x_i}{x_{i+1} - x_i}[y^{''}(x_i + 1) - \sigma^2 y(x_i + 1)] \tag{170}$$

(170) is the tension spline and σ^2 the tension parameter such that:
for $\sigma^2 = 0 \Rightarrow$ cubic spline (169)
for $\sigma^2 = \infty \Rightarrow$ linear spline
The solution of $[y^{''}(x) - \sigma^2 y(x)]$ is the *sinh*.

Bibliography

-1 Livi Bacci Massimo Popolazione e alimentazione. Saggio sulla storia demografica europea, Il Mulino, collana:Saggi, agosto 1993

-2 Roser Max, Esteban Ortiz-Ospina and Hannah Ritchie (2013) - "Life Expectancy". Published online at OurWorldInData.org.
Retrieved from: 'https://ourworldindata.org/life-expectancy' [Online Resource]

-3 Stallard E. (2006) "Demographic issues in Longevity Risk Analysis", Journal of Risk and Insurance, 73(4), 575-609

-4 Bauer, D., Börger, M. and Ruß, J. (2010). On the pricing of longevity-linked securities. Insurance Mathematics and Economics 46(1), 139–149. doi:10.1016/j.insmatheco.2009.06.005

-5 Kling, A., Richter, A., Ruÿ, J., 2008. Tax Incentives for annuitization: Direct and indirect effects. Working Paper, Ulm University.

-6 Levantesi, S. Menzietti, M., (2017) Maximum Market Price of Longevity Risk under Solvency Regimes: The Case of Solvency II, Risks 5(2):1-21

-7 EIOPA 2011, EIOPA Report on the fifth Quantitative Impact Study (QIS5) for Solvency II EIOPA TFQIS5-11/001.Frankfurt: EIOPA.

-8 Cox Samuel H., Yijia Lin, Ruilin Tian, and Jifeng Yu. Managing Capital Market and Longevity Risks in a Defined Benefit Pension Plan, Journal of Risk and Insurance 2013, 3, 585-619.

9- Li Jackie, Chong It Tan, Sixian Tang, and Jia Liu. On the optimal hedge ratio in index-based longevity risk hedging European Actuarial Journal 2019, 1-17.

10- Cowley, A. and Cummins, J. (2005). Securitization of life insurance assets and liabilities. Journal of Risk and Insurance 72(2), 193–226. https://EconPapers.repec.org/RePEc:bla:jrinsu:v:72:y:2005:i:2:p:193-226

11- Blake, D. Burrows, W. (2001): Survivor Bonds: Helping to Hedge Mortality Risk, in: Journal of Risk and Insurance, 68(2):339-348.

12- Cooper, R. W., 1972, An Historical Analysis of the Tontine Principle, Monograph 1, S. S. Huebner Foundation, The Wharton School, University of Pennsylvania, Philadelphia.

13- Jennings, R. M., and A. P. Trout, 1982, The Tontine: From the Reign of Louis XIV to the French Revolutionary Era, Monograph 12, S. S. Huebner Foundation, The Wharton School, University of Pennsylvania, Philadelphia.

14- Blake, D., Cairns, A.J.G. and Dowd, K. (2006). Living with mortality: longevity bonds and other mortality-linked securities. British Actuarial Journal 12(1), 153–197. doi:10.1017/S1357321700004736

15- Dowd, K. (2003). Survivor bonds: A comment on Blake and Burrows. Journal of Risk and Insurance, 70(2):339-348.

16- Dowd, K., Blake, D., Cairns, A.J.G. Dawson, P. (2006) Survivor swaps. Journal of Risk and Insurance, 73(1):1-17.

17- Cairns, A.J.G., Blake, D. Dowd, K. (2008). Modelling and management of mortality risk: A review. Scandinavian Actuarial Journal, 2008(2-3):79-113

18- Coughlan, G.D., Epstein, D., Sinha, A. and Honig, P. (2007). q-forwards: Derivatives for Transferring Longevity and Mortality Risk.

19- LLMA. 2010. Technical Note: The S-Forward. Washington: Life and Longevity Markets Association. October.

20- Yijia Lin, Samuel H. Cox Securitization of Mortality Risks in Life Annuities First published: 24 May 2005 https://doi.org/10.1111/j 1539-6975.2005.00122.x

21- Biffis, Enrico and Blake, David P., Mortality-Linked Securities and Derivatives (October 7, 2009). Available at SSRN: https://ssrn.com/abstract=1340409 or http://dx.doi.org/10.2139/ssrn.1340409

22- Wang, S. (2002). A Universal Framework for Pricing Financial and Insurance Risks. ASTIN Bulletin, 32(2), 213-234. doi:10.2143/AST.32.2.1027

23- Samuel H. Cox, Yijia Lin, Shaun Wang, Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization. First published: 29 November 2006 https://doi.org/10.1111/j.1539-6975.2006.00196.x

24- Denuit Michel, Pierre Devolder, and Anne-Cecile Goderniaux. Securitization of Longevity Risk: Pricing Survivor Bonds With Wang Transform in the Lee-Carter Framework, Journal of Risk Insurance 2007, 74, 87-113.

25- S. Levantesi, Menzietti, Torri, Longevity bond pricing models: an application to the italian annuity, 2010

26- Pitacco Ermanno, Matematica e Tecnica Attuariale delle Assicurazioni sulla Durata di Vita, LINT

27- Cairns AJ, Blake D, Dowd K (2006a) Pricing deaths: frameworks for the valuation and securitisation of mortality risk. ASTIN Bulletin 36:79120.

28- Lee, R.D,. and Carter, L.R (1992) Modeling and Forecasting U.S. Mortality. Journal of the American Statistical Association 87(419):671-672

29- Milevsky, M.A.,and Promislow, S.D. (2001) Mortality derivatives and the option to annuitise. Insurance: Mathematics and Economics, 29(3):299-318

30- Dahl, M. (2004) Stochastic mortality in life insurance: Market reserves and mortalitylinked insurance contracts. Insurance: Mathematics and Economics 35:113-136.

31- Biffs, E. (2005). Affine processes for dynamic mortality and actuarial valuations. Insurance: mathematics and economics 37:443-468.

32- Denuit, M. and Devolder, P. (2006). Continuous time stochastic mortality and securitization of longevity risk, Working Paper 06-02, Institut des Sciences Actuarielles, Universite' Catholique de Louvain, Louvain-la-Neuve.

33- Luciano, E. and Vigna, E. (2008) Mortality risk via affine stochastic intensities: Calibration and empirical relevance. Belgian Actuarial. Bull, 8.

34- Schrager, D. (2006) Affine stochastic mortality. Insurance: Mathematics and Economics.

35- Fadoua Zeddouk, 2020, Mean reversion in stochastic mortality:why ad how?

-36 Brouhns, N., Denuit, M. and Vermunt, J.K. (2002). A Poisson log-bilinear regression approach to the construction of projected lifetables. Insurance: Mathematics and Economics 31(3), 373–393.

-37 J. G. Cairns, David Blake, Kevin Dowd, A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration Andrew First published: 29 November 2006 https://doi.org/10.1111/j1539-6975.2006.00195.x

-38 Duffie D, Filipovic, D, SchachermayerW(2003) Affne processes and applications in finance. Annals of applied probability 984-1053.

-39 Duffie D, Pan J, Singleton K (2000) Transform analysis and asset pricing for affine jump-diffusion. Econometrica 68:1343-1376.

-40 Luciano E, Vigna E (2015) Non mean reverting affine processes for stochastic mortality ICER Working Papers.

-41 Oldrich Vasicek, Journal of Financial Economics, An equilibrium characterization of the term structure, Volume 5, Issue 2, November 1977, Pages 177-188

-42 John Hull, Alan White, Pricing Interest-Rate-Derivative Securities, The Review of Financial Studies, Volume 3, Issue 4, October 1990, Pages 573–592, https://doi.org/10.1093/rfs/3.4.573

-43 Fadoua Zeddouk and Pierre Devolder, Pricing of Longevity Derivatives and Cost of Capital Institute of Statistics, Biostatistics and Actuarial Sciences, Voie du Roman Pays 20, 1348 Louvain-La-Neuve, Belgium; Published: 15

April 2019

-44 Milevsky Moshe A.,David S. Promislow, and Virginia R. Young. Financial Valuation of Mortality Risk via the Instantaneous Sharpe Ratio: Application to Pricing Pure Endowments, Working Paper Ann Arbor: Department of Mathematics, University of Michigan 2005.

-45 Milevsky, Moshe A. The calculus of retirement income: Financial models for pension annuities and life insurance. Cambridge University Press, 2006.

-46 Cairns, A.J.G., Blake, D. and Dowd, K. (2006). A two-factor model for stochastic mortality with parameter uncertainty:theory and calibration. Journal of Risk and Insurance 73(4), 687–718. doi:10.1111/j.1539-6975.2006.00195.x

-47 Cairns Andrew J.G., David Blake, and Kevin Dowd. Pricing deaths: Frameworks for the valuation and securitisation of mortality risk, ASTIN Bulletin 2006.

-48 Biffis, E.,Denuit, M.Devolder, P. (2010). Stochasticmortality under measure changes. Scandinavian Actuarial Journal 2010(4), 284–311.

49- Barrieu, Pauline, and Henri Loubergé. "Reinsurance and securitisation of life insurance risk: the impact of regulatory constraints." Insurance: Mathematics and Economics 52.2 (2013): 135-144.

50- Dowd Kevin, David Blake, Andrew J. G. Cairns, and Paul Dawson. Survivor swaps, Journal of Risk and Insurance, 2006, 73, 1-17.

51- Loeys Jan, Nikolaos Panigirtzoglou, and Ruy M. Ribeiro. Longevity: a Market in the Making, J.P. Morgan Securities Ltd 2007.

52- Barrieu Pauline, and Luitgard A.M. Veraart. Pricing q-forward contracts: an evaluation of estimation window and pricing method under different mortality models, Scandinavian Actuarial Journal 2014, 2, 1-21.

53- Lin YiJia, Richard D. MacMinn, and Ruilin Tian. De-risking defined benefit plans Insurance: Mathematics and Economics, 2015, 63, 52,65

54- Enzo Orsingher, Luisa Beghin, Probabilità e modelli aleatori, Copyright © MMVI ARACNE editrice S.r.l. via Raffaele Garofalo, 133 A/B00173 Roma (06). II ristampa aggiornata: febbraio 2009

55- Aquilina, M., Baker, R. and Majer, T. (2017). The Money's Worth of annuities in the UK between 2006 and 2014. The Journal of the Economics of Ageing 9, 208–218. doi:10.1016/j.jeoa.2016.10.002

56- Apicella G, Dacorogna M, Lorenzo E.D, Sibillo M (2019) Improving the forecast of longevity by combining models. North American Actuarial Journal 23:298-319.

57- Barrieu, P., Bensusan, H., El Karoui, N., Hillairet, C., Loisel, S., Ravanelli, C. and Salhi, Y. (2012). Understanding, modelling and managing longevity risk: key issues and main challenges. Scandinavian Actuarial Journal 2012(3), 203–231. doi:10.1080/03461238.2010.511034

58- Blake, D. (2018). Longevity: a new asset class. Journal of Asset Management 19(5), 278–300. doi:10.1057/s41260-018-0084-9

59- Blake,D., Cairns,A.J.G., Dowd,K. and Kessler, A.R. (2018). Still living with mortality: the longevity risk transfer market after one decade. British Actuarial Journal 24(1), 1–80. doi:10.1017/S1357321718000314

60- Börger, M. (2010). Deterministic shock vs. stochastic value-at-risk—an analysis of the solvency II standard model approach to longevity risk. Blätter der DGVFM 31(2), 225–259.

61- Buhlmann, H. (1970). Mathematical models in risk theory. Berlin Heidelberg: Springer-Verlag.

62- Bühlmann, H. (1980). An economic premium principle. ASTIN Bulletin: The Journal of the IAA 11(1), 52–60.

63- Cairns, A.J.G., Blake, D., Dowd, K., Coughlan, G.D., Epstein, D., Ong, A. and Balevich, I. (2009). A quantitative comparison of stochastic mortality models using data from England and Wales and the United States. North

American Actuarial Journal 13(1), 1–35. doi:10.1080/10920277.2009.10597538

64- Chen, B., Zhang, L. and Zhao, L. (2010. On the robustness of longevity risk pricing. InsuranceMathematicsandEconomics 47(3), 358–373.

65- Coale, A.J., and Kisker, E.E. (1990). Defects in data on old-age mortality in the United States: New procedures for calculating mortality schedules and life tables at the highest ages. In Asian and Pacific Population Forum 4(1), 1–31.

66- Hull JC (2009) Options, Futures, and Other Derivatives. Pearson Prentice Hall.

67- Kac M (1949) On distributions of certain wiener functionals. Transactions of the American Mathematical Society 65:1-13.

68- Mishura Y, Posashkova S (2008) Positivity of solution of nonhomogeneous stochastic differential equation with non-Lipschitz diffusion. Theory of Stochastic Processes 14.

70- Damiano Brigo, Fabio Mercurio Interest Rate Models - Theory and Practice With Smile, Inflation and Credit Springer (2006)

71- David Lando, Credit Risk Modeling Theory and Applications
Dominic O'Kane Modelling Single-name and Multi-name Credit Derivatives (The Wiley Finance Series)Wiley (2008)

72- P.J. Schonbucher, Credit Derivatives Pricing Models Models, Pricing Implementation JOHN WILEY AND SONS LTD (2003)

73- Steven E. Shreve Stochastic Calculus for Finance II ContinuousTime Models (Springer Finance) (v. 2)Springer (2004)

74- Ralf Korn Elke Korn Gerald Kroisandt Monte Carlo Methods and Models in Finance and Insurance

75- Glasserman, Monte Carlo Methods In Financial Engineering

76- Bensusan, H., El Karoui, N., Loise, L., Salhi, Y. (2016) Partial splitting of longevity and financial risks: The longevity nominal choosing swaptions. Insurance: Mathematics and Economics Volume 68:61-72

77- Haberman, S., Kaishev, V., Millossovich, P.,Villegas, A., Baxter, S., Gaches, A., Gunnlaugsson, S. and Sison, M., (2014) Longevity basis risk: a methodology for assessing basis risk, Cass Business School and Hymans Robertson LLP, Institute and Faculty of Actuaries (IFoA) and Life and Longevity Markets Association

78- Li, J. Haberman, S. (2015). On the effectiveness of natural hedging for insurance companies and pension plans. Insurance: Mathematics and Economics. 61:286-297

79- Li, J., Li, J. S., Tan, C.I. Tickle, L., (2019) Assessing basis risk in index-based longevity swap transactions, Annals of Actuarial Science, Cambridge University Press 13(1):166-197

80- Li, J.S.H., Zhou, R. Hardy, M. (2015) A step-by-step guide to building twopopulation stochastic mortality models. Insurance: Mathematics and Economics, 63, 121-134

81- Li, N., Lee, R. (2005). Coherent Mortality Forecasts for a Group of Population: An Extension of the Lee-Carter Method. Demography. 42:575-594

82- Villegas, A.M., Haberman, S., Kaishev, V.K. and Millossovich, P.,(2017) A comparative study of two-population models for the assessment of basis risk in longevity hedges,ASTIN Bulletin, 47:631-679

83- Borger Matthias. Deterministic shock vs. stochastic value at risk : an analysis of the Solvency 2 standard model approach to longevity risk, Blatter DGVFM 2010, 31, 225-259.

84- D'Amato Valeria, Emilia Di Lorenzo, Steven Haberman, Pretty Sagoo, and Marilena Sibillo. De-risking strategy: Longevity spread buy-in Insurance: Mathematics and Economics 2018.

85- Leung Melvern, Chung F. Man, and Colin O'Hare. A comparative study of pricing approaches for longevity instruments, Insurance: Mathematics and Economics 2018, 95-116.

86- Market pricing of longevity-linked securities Sixian Tang and Jackie Li

87- Federico Guerreschi Longevity risk e tavoe di mortalità: una proposta per la popolazione italiana

88- Valeria D'Amato, Il fenomeno della longevità ed il rischio di modello: analisi e misura

89- Andrea Iliade, Previdenza complementare in italia

90- Shuo-Li Chuang and Patrick L. Brockett Modeling and Pricing Longevity Derivatives Using Stochastic Mortality Rates and the Esscher Transform

91- Hua Chen and Samuel H. Cox - Modeling Mortality with Jumps: Transitory Effects and Pricing Implication to Mortality Securitization

92- Samuel H. Cox Yijia Lin Shaun Wang, Multivariate exponential tilting and pricing implication for mortality

93- Pricing longevity derivatives via Fourier transforms Jorge M. Bravo João Pedro Vidal Nunes

94- Marco Marino, Il rischio di longevità: Tecniche e strumenti di copertura

95- Samuel Wills, Michael Sherris, Securitization, structuring and pricing of longevity risk

96- Jorge Miguel Bravo Najat El Mekkaoui de Freitas ,Valuation of longevity-linked life annuities

97- Alili L., Dufresne D., Yor M. (1997), Exponential functionals and principal values related to Brownian motion, Rev. Mat. Iberoamericana, 3—14.

98- Beghin L., Orsingher E. (1999), On the maximum of the generalized Brownian bridge, Lithuanian Mathematical Journal, 39, 2, 200-213.

99- Beghin L., Nikitin Y., Orsingher E. (2003), How the sojourn time distributions of Brownian motion are affected by different forms of conditioning, Statistics and Probability Letters, 65, 291-302.

100- Borodin A.N., Salminen P. (2002), Handbook of Brownian motion: facts and formulae. Second edition. Probability and its Applications. Birkhäuser Verlag, Basel.

101- Burrill C.W. (1972), Integration and Probability, McGraw- Hill Book Co., New York.

102- Comtet A.,Monthus C. (1996), Diffusion in a one-dimensional random medium and hyperbolic Brownian motion, Journ. Physics A, Math. Gen. 29, 1331-1345

103- Castellani, De Felice, Moriconi, Manuale di finanza I. Tassi d'interesse. Mutui e obbligazioni, il mulino, 2005

104- Castellani, De Felice, Moriconi, Manuale di finanza II. Teoria del portafoglio e mercato azionario, il mulino, 2005

105- Castellani, De Felice, Moriconi, Manuale di finanza III. Modelli stocastici e contratti derivati, il mulino, 2006

106- Rita Laura D'Ecclesia, L. Gardini, Appunti di matematica finanziaria, VII edizione, Giappichelli Editore- Torino, 2013

107- Chung K.L. (1974), A Course in Probability Theory, Secondedition. Probability and Mathematical Statistics, Vol. 21. Academic Press, New York-London.

108- Cramér H., Leadbetter M.R. (1967), Stationary and Related Stochastic Processes. Sample Function Properties and Their Applications, John Wiley and Sons, New York.

109- Doob J.L. (1984), Classical Potential Theory and Its Probabilistic

Counterpart, Springer-Verlag, Berlin.

110- Dynkin E.B., Yushkevich A.A. (1969), Markov Processes: Theorems and Problems, Plenum Press, New York.

111- Elliott R.J. (1982), Stochastic Calculus and Applications, Springer-Verlag, New York.

112- Gelfand I. M., Yaglom A. M. (1956), Integration in function spaces and its application to quantum physics. Uspehi Mat. Nauk (in Russo), 1(67), 77—114.

113- Gertsenshtein M.E., Vasiliev V.B. (1959), Waveguides with random inhomogeneities and Brownian motion in the Lobachevsky plane, Theory of Prob. Appl., 3, 391—398.

114- Getoor R.K. (1961), Infinitely divisible probabilities on the hyperbolic plane, Pacific Journ. Math., 11, 1287-1308.

115- Gihman I.I. (1972), Stochastic Differential Equations, Springer-Verlag, New York.

116- Gradshteyn I.S., Rhyzik I.M. (1994), Tables of Integrals, Series and Products, Alan Jeffrey Editor, Academic Press, London.

117- Gruet J C. (1996), Semi-groupe du mouvement brownien hyperbolique, Stochastics and Stochastics Reports, 56, 53-61.

118- Itô K., McKean H.P. Jr. (1964),Diffusion Processes and their Sample Paths, Springer-Verlag, Berlin.

119- Kannan D. (1979), An Introduction to Stochastic Processes, Elsevier North Holland, New York.

120- Karatzas I., Shreve S.E. (1991), Brownian Motion and Stochastic Calculus, 2nd Edition, Springer-Verlag, New York.

121- Karlin S., Taylor H.M. (1975), A First Course in Stochastic Processes, Second edition, Academic Press, New York- London.

122- Karlin S., Taylor H.M. (1981), A Second Course in Stochastic Processes, Academic Press, Inc., New York-London.

123- Karpelevich F.I., Tutubalin V.N., Shur M.C. (1996), Limit theorems for the composition of distributions in the Lobachevsky plane, Theory of Prob. Appl., 3, 399-401.

124- Klass M.J. (1988), A best possible improvement of Wald'sequation, Ann. Prob., 16, 2, 840-853.

125- Laha R.G., Rohatgi V.K. (1979), Probability Theory, John Wiley and Sons, New York.

126- Lamperti J. (1977), Stochastic Processes. A Survey of the Mathematical Theory, Springer-Verlag, New York.

127- Ladokhin V.I. (1962), On some random variables related to sample paths of Wiener processes, Uchenye Zapiski Kazan. Univ., 122, 4, 21-36 (in Russo).

128- Lebedev N.N. (1972), Special Functions and Their Applications,Dover Publ., Inc., New York.

129- Lévy P. (1939), Sur certains processus stochastiques homogènes, Compositio Math. 7, 283—339 (in Francese).

130- Lifshits M. A. (1995), Gaussian random functions, Mathematics and its Applications, 322. Kluwer Academic Publishers, Dordrecht.

131- Marshall A.W. (1960), A one-sided analog of Kolmogorov's inequality, Ann. Math. Stat., 31, 483-487.

132- McKean H.P. Jr. (1963), A winding problem for a resonator driven by a white noise, Journ. Math. Kyoto Univ., 2, 227-235.

133- McKean H.P. Jr. (1969), Stochastic Integrals, Academic Press, New

York.

134- McKean H.P. Jr. (1975), Applications of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov, Comm. Pure Appl. Math, 28, 323-331.

135- McKean H.P. Jr. (1976), A correction to: "Applications of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov", Comm. Pure Appl. Math, 29, 553-554.

136- Nikitin Y., Orsingher E. (2000), The intermediate arc-sinelaw, Stat. Prob. Lett., 49, 119-125.

137- Orsingher E. (1984), Elementi di Teoria della Misura e della Integrazione di Lebesgue-Stieltjes, Dip. Statistica, Probabilità e Stat. Appl., Serie C - Didattica, Univ. "La Sapienza", Roma.

138- Orsingher E. (1990), Probability law, flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws, Stoch. Proc. Appl., 34, 49-66.

139- Orsingher E. (2000), Exact joint distribution in a model of planar random motion with constant velocity, Stochastics and Stochastics Reports, 69, 1-2, 1-10.

140- Podlubny I. (1999), Fractional Differential Equations, Academic Press, San Diego.

141- Port S.C., Stone C.J. (1978), Brownian Motion and Classical Potential Theory, Academic Press, New York.

142- Prohorov Yu.V., Rozanov Yu.A. (1969), Probability Theory, Springer-Verlag, Berlin.

143- Revuz D., Yor M. (1991), Continuous Martingales and Brownian Motion, Springer-Verlag, Berlin.

144- Rogers L.C.G., Williams D. (1987), Diffusions, Markov Processes and Martingales, vol.2: Itô Calculus, Wiley, Chirchester.

145- Slepian D. (1962), The one-sided barrier problem for Gaussian noise, Bell System Tech. J., 42, 463-501.

146- Smirnov V.I. (1964), A Course of Higher Mathematics, Vol.3, Pergamon Press, Oxford.

147- TynMyint U. (1973), Partial Differential Equations of Mathematical Physics, American Elsevier Publishing Co., New York.

148- Varadhan S.R.S. (1980), Lectures on Diffusion Problems and Partial Differential Equations, Springer-Verlag, Berlin.

149- Wyss W. (1986), Fractional diffusion equation, Journal of Mathematical Physics, 27, 2782-2785.

150- Enzo Orsingher, Luisa Beghin, Introduzione alla probabilità. Dalle nozioni fondamentali alle applicazioni Copertina flessibile – 1 ottobre 2009

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