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Different guarantee types in Defined-Contribution Pension Fund

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Dedicated to
my son Alessandro

Abstract

Protection from market circumstances is a critical necessity in the context of defined contribution pension funds. This thesis is dedicated to the analysis of Portfolio Insurance strategies in defined contribution pension funds, with the aim of implementing mathematical theory for these types of protection systems. The risk to the future benefits of a defined contribution pension fund is particularly high because savings are entirely dependent on market changes. Constant Proportion Portfolio Insurance (CPPI) is a well-known example of a downside protection strategy in the literature. The CPPI is a dynamic portfolio insurance strategy that aims to protect investors against adverse market movements by guaranteeing a defined amount of money at the end of the investment horizon. As a result, the primary focus of this thesis is on presenting and evaluating CPPI applications in defined contribution pension plans, as well as investigating risks in both continuous and discrete-time trading. The application of this portfolio insurance method in pension funds introduces additional investment risks because the time horizon is often significantly longer than that of a regular financial investment. The two primary issues that develop with CPPI portfolios are the so-called cash-lock phenomena and gap-risk. The contribution of this thesis to pension fund management is the examination of cash-lock and gap-risk through the analytically and numerically computation of risk measures.

Contents

Preface

Concerns about the long-term sustainability of public pension systems have prompted supplemental pension schemes to play a growing role in the pension systems of industrialized countries. Because public pension income is smaller than income during working life, society as a whole is becoming increasingly conscious of the need to save for retirement. A growing number of professional and non-professional investors are investing in pension plans. As a result, pension plan investments have expanded, and these products are now among the most significant financial and savings products. Pension systems have been a prominent topic for researchers in recent years as a result of their fast expansion. Although pension systems are designed to improve the quality of one's financial life by providing income after retirement, many risks are associated with these systems. The uncertainty surrounding pension income, as well as the fact that the pension beneficiary is directly exposed to the financial risk of the plan portfolio, emphasize the necessity of pension fund modeling. Protection against market circumstances, in particular, has emerged as a critical issue in this environment.

The suggested thesis designs and presents various portfolio insurance solutions for managing the portfolio of defined contribution pension plans. These plans are made up of consecutive and specified premium payments[1] that are invested in the financial markets and lead to a benefit that is paid out after retirement. A capital protection mechanism is required in such pension systems to offer a minimum guarantee for fund members. We will propose a solution to this problem using various techniques of portfolio insurance with the goal of giving a minimum threshold to the amount of the savings that will be accumulated until retirement for a flow of specific contributions. Constant Proportion Portfolio Insurance (CPPI) is a common example of a strategy with downside protection. It is a dynamic portfolio insurance strategy that tries to protect the investor from adverse market movements by assuring at least a predetermined amount previously defined on the investment horizon. CPPI method was introduced by Perold (1986) [33] (see also Perold and Sharpe (1988) [34]) for fixed-income instruments and Black and Jones (1987) [7] for equity instruments; we also have to recall Black and Perold in 1992 [8] and Bertrand and Prigent's 2005 work [6]. Specifically, by altering the traditional dynamics of the CPPI for the framework of pension funds, several variants of the strategy in different markets and with different trading limitations are constructed. On a continuous and discrete basis, the portfolio efficiency of these newly introduced techniques are examined in

[1] We consider that contributions are paid monthly.

the trading markets. An important bibliographical reference for the modeling of these strategies is the article by Korn R., Selcuk-Kestel A. S., Temocin B. Z. (2017) [26] for the discrete-time treatment and the thesis by Temocin B.Z. (2015) [38] for the implementation of strategies considered in continuous-time and discrete-time trading. In a market with continuous time trading a replication technique is applied, with the goal of reducing discontinuities caused by contributions. In other words the discontinuity caused by the received contributions is removed by pricing and short-selling the claim of future payments. In this manner, incoming payments are processed as though they were already in the portfolio. Contributions are considered to constitute a fraction of labor income, which is described as a stochastic process. Then two distinct floor/guarantee procedures are specified, with the fundamental assumption that the retirement guarantee is equal to the summation of the time value of the plan participant's contributions payments. However, although offering a simple and controllable assurance system, the continuous time CPPI is unrealistic since it requires continual rebalancing, which is not the case with real discrete-time trading. As a result, this thesis focuses primarily on the scenario of discrete-time trading. We re-elaborate in discrete the components outlined for the continuous-time trading scenario, and we address the risks to which the approach is susceptible analytically (which in the continuous scenario were avoided).
The use of this sort of portfolio insurance technique to pension funds introduces new investment risks, as the investment horizon is often much longer than that of a traditional financial investment. The most common risks connected with CPPI-type techniques are thoroughly examined. Cash-lock is a significant issue that emerges in CPPI schemes with a fixed rate threshold. This is the condition in which all of the portfolio assets are fully invested in the risk-free asset and have no capacity to recoup. Because a cash-locked position limits any participation in a market gain, it is seen as a key risk, particularly for long-term investments. Balder and Mahayni [4] estimated cash-lock probability for several portfolio insurance methods, including traditional CPPI, in continuous time and presented the performance comparison results. The Gap-risk is another significant risk that affects a CPPI portfolio. This is the probability that the portfolio value will fall below the floor level and fail to guarantee the required amount at maturity. Balder S., Brandl M., Mahayni A. (2009) [3] investigated the problem under the assumption of discrete-time trading and presented risk measures for quantifying the gap-risk for a CPPI with a fixed-growth floor. Hedging techniques utilizing artificial assets to represent price jumps and price gap risk are also discussed in the literature. Tankov provides unusual gap options in [37] to hedge against gap occurrences, or jumps in an Lévy-type framework. A additional issue emerges in the event of a fast gain in the market, when the minimum value (the bond floor) becomes tiny in comparison to the portfolio's value.[2] The fundamental issue is that, while the portfolio's value improves greatly in increasing markets due to a big number of shock absorbers, the potential loss is high even if the minimum level stays low. In relation to the risks mentioned above, we will calculate the relative risk measures, i.e. the Cash-Lock probability and the Shortfall probability, as well as the Expected Shortfall that should support the pension fund if

[2]Include a ratchet mechanism in the floor since this may result in a huge potential loss and prohibit you from taking benefit of the increasing market.

the fund falls below the minimum guarantee (bond floor). The period under consideration will be split into n instants, and the risk measurements are evaluated in each instant. These measurements are obtained in closed form if computed just before the periodic payment of the contribution; they are obtained as an approximation if the increment due to the payment of contribution is considered. This distinction is important because the positive amount of the contribution paid could raise a situation of cash-lock or shortfall. Finally, we proceed with a numerical application, in which we discuss and compare the effectiveness of each strategy through Monte Carlo simulations. We provide a detailed analysis of the CPPI strategy's behavior with an NPV floor and a random floor. We study the distributions of portfolio's value at maturity and the performance of both strategies based on the evolution of risk measures. The sensitivity of strategies to changes in multiplier is also assessed.

The paper is organized as follows:
The first chapter will be dedicated to presenting the legislature and the functioning of pension funds, with specific focus on the phase of contribution accumulation and, as a result, the applicable investment strategies. Following in the second chapter a brief introduction to Portfolio Insurance, we will deepen the constant proportion strategies and the risks associated with them. Outside of the context of pension funds, the CPPI will be displayed in the standard version (in continuous time), and then the trading time will be discretized. The third chapter will be devoted to modeling the above-mentioned defined contribution system as well as the parts of the pension fund portfolio management technique. CPPI methods in continuous time trading with different random dynamics are introduced. The floor processes in the designed continuous environment are classified as net present value (NPV) floor, which is the time zero value of the future payments claim, and random floor, which is the time value of past and future payments. The same topic is treated in a more realistic context with discrete-time trading to further assess the risks avoided by continuous trading. We redefined the NPV and the random floor to retain their discrete structure. Cash-lock and gap-risk are handled, and related risk measures are computed.The performance of these techniques is analyzed using terminal wealth distributions and sensitivity analysis for the risk measures. Finally, we present the conclusions as well as prospective future developments of the strategies discussed. Please see the final appendices for further information on some basic topics employed in the thesis' development.

Chapter 1

Pension Plan

The Italian social security system is built on a three-pillar framework:
Pillar I concerning with the public system, in which all residents, whether private employees, public employees, or self-employed, are required to participate in the state pension system.
Pillar II is based on voluntary participation in *collective* forms of pension funds, which are primarily connected to employment status, with the goal of integrating the benefits given by the basic system and ensuring an appropriate quality of living after retirement.
Pillar III concern the individual form of pensions. Each person has the freedom to join a pension fund or to get into a pension insurance contract.
Pillar II and *Pillar III* represent complementary pension systems with collective and individual adherence, respectively.
The public social security system in Italy is a mandatory *pay-as-you-go* system, which means that the contributions made by employees in one calendar year are used to pay the payments of retirees in the same calendar year. It is apparent how such a system is inextricably tied to the demographic structure of its believers as well as the work level of the country's residents.

In the 1980s, Italian pension system enters into crisis, the reasons are attributable to causes of a demographic nature: aging of the population, due to the lengthening of the average life span and the decline in births; causes of an economic nature: increase in unemployment; shorter active phases and longer retirement periods; "generosity" of the services provided by the system social security.

In particular on the extreme generosity of the payments we can identified three major aspects:

- Almost all benefit expenditures were marked by high replacement rates [1] and pension indexation. The availability of an early retirement option with no actuarial penalty reinforced this tendency even further.
- A widespread aspect of redistribution. This was due to the cohabitation of a number of separate systems, each with its own set of ad hoc regulations.

[1] The replacement rate is the ratio between first pension installment and last salary; it provides a measure of how retirement replaces income and ability of the system to guarantee the worker a similar standard of living a that conducted during work.

To establish political consensus, policymakers used unequal treatment among funds extensively.

- A continually substantial deficit between payroll taxes and expenditures, indicating a systemic financial disequilibrium.

All of these considerations, together with rising concerns about the long-term sustainability of public pension systems, have led complementary pension schemes becoming far more relevant.
Because public pension earnings are lower than those earned during working life, society as a whole is becoming more conscious of the need to save for retirement. Pension plans are being hired by an increasing number of professional and non-professional investors. As a result of this, investments in pension plans have increased, and these products now play a significant role in the global fund sector. In industrialized economies, pension plans are one of the most important financial and savings products.
Italy has enacted important reforms to public pensions, including the introduction of hybrid systems to establish a more sustainable pension system and a focus on pension plan investment. The reforms implemented during the 1990s tried to tackle a series of problems inherent in the previous system.

Let us now outline the most important reforms that led to the introduction of supplementary pension.

1.1 Regulatory references

- Legislative Decree n. 124/1993 (*Amato Reform*).
  It is the first reform that aims to rationalize supplementary pensions and defines its main characteristics:
  - Voluntary membership
  - Recipients: private and public employees of the different categories, self-employed and freelancers
  - Establishing the institutive sources: collective agreements, agreements, agreements among self-employed workers, or promoted by companies or trade unions.
  - Governance: constitution and definition of control bodies of the fund
  - Operating rules, methods and entities authorized to manage of resources
  - Rules concerning financing and benefits

  The decree establishes the authority responsible for the supervision of complementary pension forms, COVIP. It also introduces tax benefits to promote adherence to supplementary pensions. The Amato reform imposed strict regulations on the supplementary pension system; however, despite high expectations that supplementary pensions would be a tool for overcoming sharp contractions in the public pension system, it essentially discovered a lack of adhesion and development of pension funds.

- Law n. 335/1995 (*Dini Law*)
  The most important aspects of intervention of the Dini Law on supplementary pension can be summarized as follows:
  - Expansion of recipients, institutional sources and methods of participation
  - Portability, which allows a worker to switch from one type of social security to another, provided that the regulatory conditions are satisfied (the possibility of mobility between pension funds is established after 5 years from constitution, or 3 years if fully operational)
  - New rules on tax and contribution treatment:
    Abolition of the 15% tax on contributions for pension funds cashed
    Deductibility of contributions paid for employers and recipients
    Incentive to businesses for the allocation of severance pay to finance pension funds
  - Expansion of the operations of pension funds allowing to undertake insurance-type commitments for the payment of annuities
- Ministerial Decrees n. 673/1996 and n. 703/1996.
  The first aforementioned decree govern the criteria and methods for the management of resources, which must take place by asset management companies. Resource investment criteria and limits, as well as rules on conflict of interest, are established in second decree:
  - Diversification of investments (debt / equity securities, UCITS, closed-end funds)
  - Efficient portfolio management: use of derivative instruments
  - Diversification of risks, including counterparty risks
  - Containment of the transaction, management and operation costs of the bottom
  - Maximization of net returns
- Delegation law 243/2004 *Maroni Reform* and Legislative Decree n. 252/2005
  To incentivize the adhesions to pensions funds, the reforms aim to make economic discipline more favorable.
  One of the main objectives is to increase funding flows to complementary pensions through the granting of the accruing severance pay (TFR[2]). It is possible for each worker's future pension to be financed through his or her TFR. Prior to 2005, a portion of supplementary pension fund contributions was already compounded by a component of the TFR, however the decision to

[2]The severance pay (or post-employment benefit) is know in italy as *Trattamento di Fine Rapporto*, that is TFR. It can be defined as an annual payment of a portion of an employee's salary.

join was not totally independent of the company's will. The new *silent-consent* system was designed to allow workers to automatically and directly confer the TFR as a sort of additional pension. The COVIP's powers have been increased to include ensuring appropriate disclosure, correctness of behavior, sound and responsible administration of pension funds, and, more broadly, ensuring the effective operation of the social security system. The Pension Fund Supervisory Commission[3] oversees the orderly development of the supplementary pension system with broad powers and prerogatives of supervision, inspection, sanctions and regulations.[4]

Although the outcomes and long-term impacts of these reforms have been uneven, some of them, such as the 1993 reform, have resulted in significant reductions in government spending. Others, on the other hand, were not as effective since they failed to include population projections for the next 30 to 40 years.Nonetheless, the 1995 reform resulted in a significant reduction in pension spending by introducing a hybrid system that is partly PAYG and partly financed, with a focus on supplementary social insurance programs. Despite this, the system is not yet complete, as the complementing social insurance measures have been delayed. All of these changes not only assist to reduce pension spending and balance Italian state finances, but they also result in a major reduction in social coverage levels.

1.2 Type of supplementary pension schemes

Now we briefly outline the types of pension funds provided for by the Italian legislation. We can recognizes two broad categories of complementary pensions: occupational plans and personal plans. There are four categories of pension plans: negotiable (closed or contractual), open, and pre-existing, which belong to the first category, and individual pension plans, which belong to the second group.
Let's take a closer look at each type.

Occupational plans

- *Negotiable pension funds* are created through national or corporate collective agreements. They are non-profit organizations formed as a result of a collective bargaining agreement or an agreement between employees organized by trade unions or groups. These are divided into three categories based on the breadth of membership:
    - Individual firms or groupings of enterprises form corporate or group plans.

[3]COVIP is the Italian acronym for *Commisione di Vigilanza sui fondi Pensione*, in english we can translate this as Pension Fund Supervisory Commission

[4]The COVIP's powers include: authorizing pension funds to practice their profession and maintaining a register of authorized pension funds; approval of company statutes, fund rules, and verification of the adequacy of the organizational structure; assurance of proper fund management, both in the accumulation and payment phases; and definition of disclosure schemes to ensure transparency.

- Sub-category plans are designed for specific groups (reference sector) of people or industries.
- Territorial plans: regional groupings were formed.

According to the internal guidelines, participation is limited in terms of reference industry or firm.
The legal form is the associative form outlined by art. 36 civil code (c.c.)[5] or art. 12 c.c.[6] The process that closed pension fund has to go through in order to obtain the authorization to operate is as follows: formation of the founding agreement and drafting of the statute/regulation of the fund; which must follow the notarial deed of incorporation and submission of the request to COVIP. Once the authorization is obtained, the registration with Register of Funds begins and the operation of fund commence.
Pension funds with legal subjectivity shall have an effective system of government that ensures the sound and prudent management of their business. That system shall provide for a transparent and appropriate organizational structure, with a clear allocation and appropriate separation of responsibilities and an effective system to ensure the transmission of information. The governance system shall be proportionate to the size, nature, scale and complexity of the pension fund's activities. The governance system is described in a dedicated document and takes into account related environmental, social and corporate governance factors in investment decisions. The document shall be drawn up on an annual basis by the administrative organ and shall be made public together with the budget. The governance bodies are:

- Board of Directors: It is composed of representatives of employers and employees. All members shall be of good repute and professionalism. The administrative organ of the form pension[7]:
  1. defines and adopts the investment suitable for the achievement of strategic objectives and shall monitor compliance with them; and this purpose shall examine the management reports and assess the proposals made the financial function, and recommendations of the Financial Committees and the advisor (where present), adopting the its determinations;
  2. decides the custody and the revocation of management mandates or, in the case of direct management, identifies the subjectsin charge of management;
  3. periodically revise and amend if the necessary investment policy;
  4. exercises control over the activity carried out from the finance function, assuming the relative determinations;

[5] The internal system and the administration of associations *not recognized* as legal persons are governed by the agreements of members. The said associations may be sued in the person of those to whom, according to these agreements, the presidency or direction is conferred.

[6] Private associations, foundations and other institutions acquire legal personality through the *recognition* granted by decree of the President of the Republic. For certain categories of entities that carry out their activities within the province, the Government may delegate to the prefects the power to recognize them by their decree.

[7] From COVIP- Resolution 16/3/2012 - art. 4

5. approves internal control procedures financial management, taking into account the proposals made by the function finance;
6. defines the strategy for exercise of the fund's voting rights;

– Board of Statutory Auditors: as in the Board of Directors, members must partly represent employers partly employees. The role of Board of Statutory Auditors is crucial when approving the financial statements.
– Fund manager certifies that the pension form is managed in their sole interest, in accordance with applicable legislation, and in accordance with the conditions of the regulations and contracts. shall also promptly notify the company's administrative and supervisory bodies of any anomalies discovered, indicating the corrective actions to be taken. . The person in charge must prepare an annual report on the control processes used, his organization, the results of his work, any abnormalities discovered, and the steps taken to correct them. The COVIP, as well as the administrative and supervisory body, will receive the report. It bears a great deal of responsibility to its members, thus it must maintain the highest standards of integrity, professionalism, autonomy, and independence.

According to the legislation, the pension fund is prohibited from managing investments and covering the risk of longevity at the time of distribution of social security benefits. The management of financial resources (which we will analyze extensively in the course elaborated) must be delegated through a management agreement to Banks, brokerage companies, Insurance companies, savings management companies. The insurance management is instead entrusted[8] through an insurance agreement to an insurance company that will handle the payment phase (decumulation). administrative management is also outsourced: as regards accounting, administration individual positions and so on. It is mandatory by law to delegate an appropriate Depositary Bank for the holding of accounts and custody of securities, and for the Financial contribution reconciliation and investment control procedures. The depositary bank must be an independent entity with no ties to the sources of the negotiating fund's establishment. The legislator has established this requirement to avoid the conflict of interest that would arise between the members of the board of directors and the members of the fund.

- *Open pension funds*: these plans allow all sorts of workers (employees and self-employed) to participate, even if they are not covered by a collective bargaining agreement. Therefore these funds are intended for a wide audience of people who can join both in form individual and collective (collective agreement); They are not constrained in terms of reference category or region by definition. Banks, brokerage firms, insurance firms, and asset management firms are the institutional institutions for this sort of fund.
  According to art. 2117 c.c.[9], open pension funds are constituted with separa-

[8]subject to specific exceptions

[9]The special funds for social security and assistance which the employer has set up, even without contributions from employers, may not be distracted from the purpose for which they are intended and may not be enforced by the creditors of the contractor or the provider of employment

tion of accounts and assets. It has been established that assets put aside for charitable or social security purposes are directly safeguarded by third-party creditors of the entity that formed them. The assignment constraint causes what is formally known as asset segregation inside the institution's assets.
Obtaining initial authorization to operate necessitates decisions from both the board of directors and the shareholders' meeting. As a result, a regulation is established, and a request is sent to COVIP, who then handles the authorization. The pension fund is registered in the Register of Funds and can begin operations after the board of directors makes a decision.
A representative body has been added to the previously exposed governance bodies. In the case of collective accessions involving at least 500 workers belonging to a single company or group, a representative body must be set up composed of a designated representative the same holding or group and a representative of the workers, for each communities. The representative body shall act as liaison between the communities who adhere to the fund and the company that manages the open pension fund and the manager.
As regards the management of financial resources, the open pension fund has the opportunity to manage its investments directly or to outsource this function. In the case of outsourcing, the above applies to negotiated pension funds. The insurance management can be direct or indirect, then outsourced to insurance companies through an insurance agreement. For the depository bank the above is valid.

- *Pre-existing pension funds*: they were in existence before to the 1990s changes. They are classified as occupational plans since they are based on collective agreements. They are now undergoing a process of rationalization and simplification through dissolution and merger activities.

Personal plans

- *Individual pension forms, or personal pension plans via insurance plans*[10] (D.lgs. 47/2000) can be taken out by anybody. They are a collection of numerous individual forms that may be used to augment a worker's pension. PIP are often carried out through life insurance contracts connected to class I or III such as unit-linked products or with-profit contracts.
Individual pension plans are available to a wide range of people who can join on their own (self-employed and employees; each category of entities). Insurance firms are institutional entities. Consequently, the legal form taken by the Pips are insurance contracts (art. 1882 c.c[11]) with use of contributions in Separate Management/Internal Funds, established pursuant to art. 2117 of the C.C., with accounting and patrimonial separation. Unit connected (multicomparty)

[10]Knows as PIP (*Piani Individuali Pensionistici*) in Italian

[11]Insurance is the contract under which the insurer, in payment of a premium, undertakes to claim against the insured, within the agreed limits, the damage caused to him by an accident, or to pay a capital or annuity upon the occurrence of an event pertaining to human life

vs. revaluable at repeating single premiums is the technical form. IVASS and COVIP will jointly control individual pension funds.

- IVASS[12] (Insurance Supervision Institute) deals with the insurance aspect of the pension product.
- COVIP oversees the correct application of tax and social security legislation.

The Board of Directors, the Board of Statutory Auditors, and the Head of plan are the governance bodies, just as they are for other types of pensions. The governance bodies shall coincide with the bodies of institution that generated the pension plans. Individual Pension Plan is directly involved in financial resource management, insurance management, and administrative administration of the retirement plan (direct management).

Another distinction is the mechanism used to determine contribution rates.

The system may be set up in two types of pension forms controlled by a funded scheme: *Defined Benefit* and *Defined Contribution* pension forms.

- The Defined Benefit (DB) is a method in which the level of the pension that the worker will receive is guaranteed at the moment of subscription. The pensioner's benefit is usually related to particular factors like a percentage (*Replacement Rate*) of the previous wage or the amount of state pension earned. The necessary contributions are calculated using specific assumptions to discount future contributions and benefits to ensure that the fund remains in balance. These assumptions can include investment returns, salary dynamics, demographic mortality trends, and so on. It is evident that not only demographic and financial assumptions must be made, but also a wage dynamics hypothesis, as it will be essential to start with an estimate of the latest income, which is not known at time of subscription. Therefore in this approach, pricing is based on a set of actuarial assumptions, and any departures from these assumptions would harm the fund in the long term. What has just been mentioned highlights how much of the risk in a defined benefit pension fund is borne by the fund manager.
- In the Defined Contribution (DC) case the contribution rates that the worker pays are fixed in advance (typically a portion of the member salary income or a predetermined amount), and the benefit is then computed as a result of the financial management performance that the pension fund is able to obtain on the pool of contributions invested, the accumulation period and the actual mortality rate of the insured population. As a result, it is clear that a certain level of benefit is not guaranteed at the time of retirement in this system.[13] The fund's risk exposure is reduced, and some of the risk is transferred to the members.

[12]IVASS: Istituto di Vigilanza sulla Assicurazioni (in italian)
[13]Unless the underwriter offers a capital protection or a death benefit

1.3 The management phases of a complementary pension scheme

Participation The first stage is, of course, the *participation* of workers in a pension fund. As mentioned above, membership of the supplementary pension scheme is always voluntary. Unlike joining the public pension system, which is mandatory.

Contribution We proceed with the *contribution* phase. There are no restrictions on the level of contributions that the member can pay, except limits of tax deductibility.
The contribution consists of several elements:

- employer's contribution
- worker's contribution
- annual allocation to the TFR

According to Legislative Decree 252, if the annual Report Termination Treatment shares are allocated to a pension fund, the latter guarantees the member the paid-up capital as TFR. This creates a financial commitment for the pension funds of the recipients.
The organizational model of the supplementary pension is the Defined Contribution (DC) system set out in the previous paragraph. Logic of Single Recurring Premium of Life Insurance contracts shall be able to describe the manner of organization of a defined contribution pension fund. Each payment to the social security program (contribution) activates an investment line, which is characterized, in the case of compartments secured, by one or more financial guarantees. As a result, each applicant's single premium represents a share of the benefit obtained under a predetermined deposit plan. The contribution phase is based around the activation of as many individual pension accounts as there are fund members. From an administrative perspective, at the moment of participation, a specific current account for the member is activated, on which contributions are sent on a regular basis according to the three sources of funding identified, and from this is taken what is required to carry out the investments time by time. The assessment of the social security position of the member has the contributions and the investment allowance in the accounts. It can be easily noted that in this context there is a reference to the logic-management system of type insurance unit linked to recurring single premiums.

Accumulation Then follows the phase of *accumulation*. In accordance with the provisions of the pension fund, members shall pay the contributions that will be invested by the pension fund in accordance with the criteria of sound and prudent management. Members pay ab front commissions (levied from contributions) and commissions running, which are management commissions imposed year by year on

the amount of assets managed. The first are commissions that cover and finance the annual costs of administrative management of the social security position, whereas the second (also known as management fees) are commissions deducted from the fund's assets and intended to cover financial management expenses and, in the case of financial guarantees, to pay the cost of the guarantee provided by third parties to the pension fund.
The financial system for managing a supplementary pension scheme is *individual capitalization*(or *fully founded system*).
Capitalized financial management systems shall be based on the principle of establishing the actuarial balance between average present value of contributions and average present value of liabilities. The worker's contributions are set aside (DC system) and managed in this situation with the goal of increasing the accumulated capital and ensuring the future pension. As a consequence, there is an instantaneous connection between the quantity of contributions, the outcome of their administration, and the retirement benefits. The sums paid by the assets are invested in the market, but at the same time fund must have the reserves that attest to the commitments made to the members. Throughout life of worker there will be a reserve both during the active phase (in which the contributions converge and there is a accumulation of reserve) and during a phase of payment of the benefit (where this reserve exists but is gradually dismantled).
Capitalization schemes includes:

- *Individual capitalization* for homogeneous risks, which is carried out with the fair premiums of free life insurance, since they are treated as persons of the same age, sex, and so on.

- *Collective capitalization*, on the other hand, also called heterogeneous risk capitalization, is defined by various balancing premiums, such as the general average premium or the average premium per generation. In this case, the social security charges, which will be generated by the entire collective or by the individual generation, will be spread over the whole collective, resulting in a transfer of sums from one insurance position to another, according to a principle of mutual insurance, not only of individual-funded forms, but also of insurance solidarity. The transfer is made between all members of the reference community, regardless of which risk class they belong to, as is the case with individual capitalization.

Let us explain what we mean by *sound and prudent management principle.*
We recall the Ministerial Decrees n. 673/1996 and n. 703/1996, to which we add the most recent decree n. 166/2014. The latter, compared to the 1996 Decree, advocates greater flexibility in the operator's investment choices. The 1996 Decrees imposed restrictions and detailed constraints on types, issuers and financial instruments at the expense of management efficiency. As a result, 2014 decree is pushing for greater accountability of the Board of Directors and an increase in internal control.
We examine the topics on which the 2014 decree concentrates.
The aim is to optimize the combination of profitability and risk. In the first place, the portfolio is required to meet the criteria of quality and liquidation of investments taking into account liabilities. Diversification of the portfolio is clearly required

to control risk. To minimize concentration, it is necessary to avoid focusing the majority of investments in a single issuer or within the same group of companies. It's important to distinguish between different sector of activity and geographical areas.
The risk profile and temporal structure of liabilities must be compatible with investment strategies (which we will go over in depth in the following chapter). This is an essential principle of efficient management. For the reduction of investment risk and the realization of efficient management it is possible to use derivatives (transactions with derivatives are allowed only for the aforementioned purposes). it is necessary to assess the risks associated with the operation of derivatives by monitoring the exposure generated by the use of these instruments. The level of risk is checked against the replicating financial portfolio of the derivative. As previously stated, Decree 166/2014 encourages greater financial management flexibility.On the other hand, this implies a more structured philosophy of risk control and monitoring. Ex ante monitoring and ex post verifications of management results should be established using appropriate methods and technical frameworks. Stress testing is required to ensure the fund's solvency and that the benchmarks are consistent with the fund's investment objectives and policies.

Various investment profiles, corresponding to various sectors, are available through supplemental pension systems. Normally, the assets of a fund are structured in several compartments from the perspective of heritage organization and the need to be able to capture the risk profiles of members. Each compartment reflects the fraction of assets under management whose strategic allocation is designed in such a way that the profile can be identified. As a result, we're dealing with a multi-component capital structure (to look at it another way, unit linked).
For completeness, we identify the various sorts of compartments from which a fund subscriber might choose based on his risk appetite.

- *Monetary sector* is made up of liquid financial assets with a short investment horizon. It's sector with an investment profile not exceeding one-and-a-half years and therefore consisting of government bonds. This ensures that assets are liquidated quickly and at a low cost of return. This industry has an investment strategy that relates to a benchmnark focused on capital preservation. In most cases, this is accomplished by Bot rollover methods. As a result, the pension fund will be able to track any short-term swings affecting the term structure of interest rates in real time. This compartment has a low risk profile.

- *Pure bond sector* consists of bond investments with a time horizon of less than 5 years and a medium-low risk profile. Typically, the bond sector is subject to interest rate risk, as changes in value are related to unexpected changes in the maturity curve and are reflected in the prices of the assets.

- *Mixed bond sector* primarily made up of bond investments (about 80%-90%), with an investment horizon of 5-7 years and a medium-low risk profile.

- *Equity sector* comprises of more than 50% equity investments, an investment horizon of more than or equal to 10 years, and a risk profile of medium-high - high.

- *Balanced sector* consists of both bond and equity investments (with the latter accounting for less than 30% of the total), with an investment horizon of 7-9 years and a medium risk profile. We aim to create wealth, taking advantage of the effects of negative correlations which they are established on the markets between shares and bonds.

- *Guaranteed sector* consists of 95% bond investments and 5% equity investments; investment horizon is typically less than 5 years; risk profile is absent because there is a guarantee that if the fund's returns on investments do not cover the shortfall, the fund will have to make up the difference.
  This is the sector to which, with the mechanism of silence assent, the contribution rate deriving from the TFR is allocated. Given the law, the sector that receives the TFR allowances must guarantee the restitution, at the time of the request, at least the contributions paid (hence their nominal value).[14] Management style, therefore, must replicate with the best approximation the revaluation recognized by law to the TFR is possible. All this means that these sectors have an additional burden on recognition of this guarantee and this burden is carried out through the determination of a management fee, applied to the amount of assets, which is called Commission for Guarantee. This represents in actuarial terms the periodic premium that the fund pays to an insurance company on life, which assumes the burden of integration of assets for the purposes of satisfaction of guarantee itself. Thus, a true and own transfer of financial risk, it being understood that such transfer is not complete, being that it remains at the bottom pension counterparty risk (it should be noted that the collateral even if transferred to a third party is offered by the pension scheme that therefore assumes all responsibility for its own adherents).

Let's have a look at the many types of guarantees that a fund can or must satisfy. We define the contractual guarantees in a supplementary pension scheme:

- *Contractual guarantee in the strict sense*: it is a contractual obligation (guarantee legal capital) assumed by the pension fund, of return of the greater value between the change of the Net Asset Value (NAV)[15]and the value of the minimum guarantee when the deadlines set by the pension fund regulation are reached.

- *Moral guarantee*: is a reputational obligation assumed by the pension fund to ensure the financial result in probability (sector with comparable yield target the revaluation of the TFR). The guarantee is excluded in the case of implementation of extreme events.

The contractual guarantee is normally guaranteed by a third party (guarantor) which contractually assumes financial risk.

Financial guarantees offered by a pension fund can be divided into two categories:

[14]There is therefore a guarantee with a guaranteed minimum rate of 0%.

[15]NAV represents the valorization of all financial assets invested by of the pension fund/compartment, net of the tax burden on it, to some the reference date

- *Upstream financial guarantees*: it specifies a maturity for calculating the payoff, and that maturity is attributable to the whole audience of members in the guaranteed compartment. That is, assuming the upstream guarantee is offered over a five-year period, all members who are in the guaranteed compartment at time 0 (activation time of the guarantee) will be protected from any decline in the value of their social security position in relation to the pension fund's invested capital at the end of the five-year period.

- *Event financial guarantees*: it is still a protection offered by the fund pensions to members participating in the guaranteed sector, but shall take place if and only if the typical events covered (retirement due to old age or old age, retirement due to invalidity, or retirement on death of the worker and therefore survivor's pension) occur. The difference between a private pension fund and a public pension scheme is that during the accumulation period, the member of pension fund can choose who will receive the benefit in case of death, rather than having to assign the benefits to spouse.

Furthermore, there is the possibility of inserting several minimum guarantees. For example, to establish that the accumulated value at maturity is not lower than the accrued value calculated at a certain guaranteed minimum annual rate, or that the accumulated value is never degressive over time.

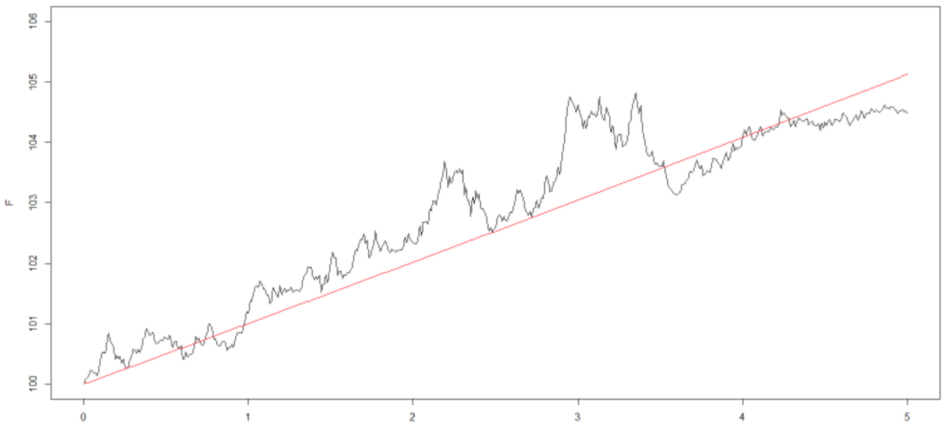

Figure 1.1. The value of a five-year-maturity fund has been modelled. The beginning value is 100, with a minimum assured rate of 0.01 taken into account. The fund's value at maturity is lower than the value gained by using the guarantee.

Figure 1.1 shows the results of a simulation conducted using the first guarantee described. The red line depicts the growth of the accumulated value at a guaranteed minimum annual rate of 0.01. Instead, in Figure 1.2, we see a possible case in which the second guarantee is used. The red line in the graph, once again, describes the guarantee.

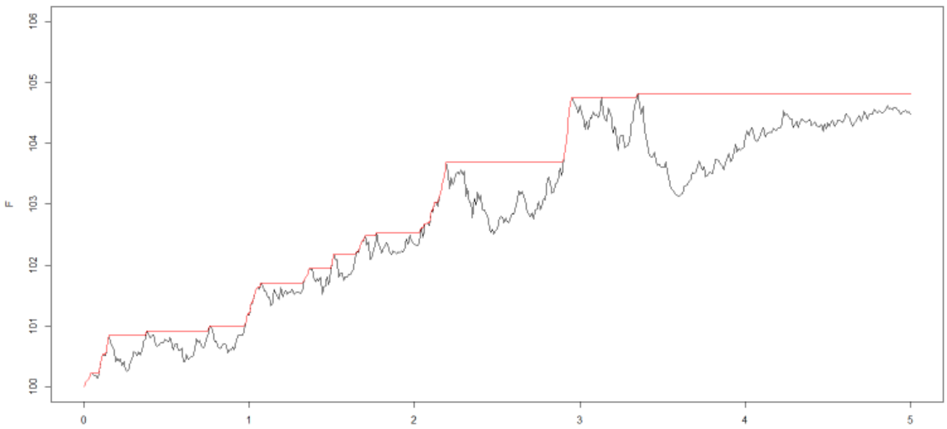

Figure 1.2. The value of a five-year-maturity fund has been modelled. The beginning value is 100. Also in this simulation the fund's value at maturity is lower than the value gained by using the guarantee.

The two guarantees can be described from an analytic-structural standpoint in the following way :

- *Expiry Guarantee* (best-off) provides a final value based on the product of mono-periodical fluctuations in the unit value of the quota (compensatory effect of positive and negative variations in the previous period).

$$Y_{k,n} = D_k \max\left\{ \frac{I_n}{I_k} M_{k,n} \right\} = D_k \max\left\{ \prod_{h=k+1}^{n} \frac{I_h}{I_{h-1}}, M_{k,n} \right\}$$

  Only at maturity the control is activated. (It is necessary to duplicate the reward of a European-style option.)

- *Cliquet Guarantee*: allows for the time-to-time consolidation of the guarantee, as well as the control of the guaranteed minimum over each unit of time. As a result, it is more expensive than the maturity guarantee.

$$Y_{k,n} = D_k \prod_{h=k+1}^{n} \max\left\{ \frac{I_h}{I_{h-1}}, M_{h-1,h} \right\}$$

Where:

- $Y_{n,k}$: value of social security benefit at time n at recurrence k, activated with payment of contribution D_k;
- D_k: contribution paid to the k-th recurrence;
- I_k: the fund's share unit value at the general recurrence k;
- I_n: NAV value in n. This is an appropriate combination linear of a risky component (shares) and one not risky (bonds);
- $M_{k,n}$: is the minimum guaranteed accumulation factor. The maximum between a set of reference indices or metrics can be used to express the minimum guarantee level.

Decumulation The last phase of a supplementary pension scheme is the payment of benefits, the *decumulation.* Purpose of a pension fund is to provide additional benefits in addition to the basic one. As a result, after the member will have access to the public pension and has completed at least five years of participation in supplementary pension plans, he or she will be eligible for the pension.
Depending on the member's preferences, the pension may be paid in the following forms:

- an annuity, which may be reversible, paid periodically (with the possibility of specific winding-up options)
- capital not exceeding 50% of the total accrued on the individual pension position and the conversion of the remainder into an annuity.

The benefit provided by the pension fund can therefore cover all the events protected by our compulsory system, not only the pension that is either old-age or anticipated. These normally include benefits for other causes, the most important of which are invalidity or death. As far as invalidity is concerned, the fund may grant a pension to an invalid or incapacitated worker, provided that specific requirements laid down by law are laid down. A person who, because of an infirmity or physical or mental defect, finds himself in the absolute impossibility of carrying out any work activity is defined as unfit. An invalid is defined as a worker whose capacity to work is reduced by a physical or mental defect to less than one third. Incapacity is permanent and invalidity is temporary; However, a worker who is found to be invalid is checked every three years to certify his invalidity and if the third check is still invalid then his position becomes that of incapacity. As far as the death event is concerned, the fund may provide for a survivor's pension if the worker was already retired at the time of death. This will be a percentage of the value that the pensioner received when he was alive and will depend on the surviving family. The survivor's pension is called an indirect pension if the cause was still a worker at the time of death. In this case the share of the pension is not known and you will have to rebuild its amount and then convert it into that of survivors.
The legislation also provides that you may apply for non-pension benefits for certain reasons.
After eight years of accession, an advance payment may be required for:

- Purchase or renovation of the first home per se or for children up to a maximum of 75% of the individual position
- additional requirements of the member up to 30% of the individual position

It may also request an advance of the position at any time for medical expenses for treatment and extraordinary interventions, proven by the competent public authorities, for if and for members of your household up to 75% In addition you can redeem or transfer the position accrued due to:

- transfer by change in work activity (100% of position at any time)
- transfer by individual choice (100% of position with minimum 2 years of membership)

it is also possible to:

- redemption in the event of cessation of work (50% of the position if unemployed for more than 12 months)
- total redemption of the position if unemployed for more than 48 months
- full redemption in the event of collective membership, for loss of participation requirements

1.4 Financial strategies for the management of social security assets

We covered the phase of contribution accumulation in the previous section; now we'll dig deeper into the method used by the fund to manage its resources and control the risks connected with management.

First we outline the regulatory evolution of the sector for risk control and investment policy:

- COVIP- Resolution 16/3/2012 - art. 3:
  1. indicates the objectives which pension form aims to achieve with reference both to the overall activity and to that of the individual compartments.
  2. The final objective of the investment is to pursue efficient risk-performance combinations in a given time span, consistent with the benefits to be provided; they should maximize the resources intended for performance by exposing members at a level of risk deemed acceptable.
  3. For the achievement of the final objective, the pension form defines the number of which it considers it useful to put in place, the risk-return combinations thereof,the possible presence of life-cycle mechanisms and its operation. To this end they shall carefully analyzed the socio-demographic characteristics of the reference population and his social security needs.
  4. For each compartment it must be made explicit the expected average annual yield and its variability in the time horizon of management. The latter must be expressed in number of years. The yield must be expressed in real terms. In the case of compartments with a fixed time horizon (e.g. target dates), the financial objective should be periodically revised according to the residual time span. The probability that based on past experience, performance of the investment, over the horizon management, is less than a given boundary.

- Ministry of Economy and Finance - Pension Funds - Decree of 7 December 2012 - n. 259:

  Art. 4. (*Technical provisions*):

  1. Pension funds shall constitute technical provisions adapted to the financial commitments made in respect of members in employment, pensioners and beneficiaries at all times sufficient assets to cover.
  2. The calculation of technical provisions shall be carried out and certified by an actuary and carried out every year. [...]
  3. Technical provisions shall be defined in compliance with the following principles:
     (a) the minimum amount shall be calculated on an individual basis taking into account the members of the fund at the date of assessment, using a method sufficiently prudent prospective actuarial, taking into account all the commitments for benefits and contributions in accordance with pension discipline of the pension fund. It shall ensure that disbursement continues beneficiaries of pensions and other benefits whose enjoyment has already begun and allows to meet the commitments arising rights already accrued by members;
     (b) economic, demographic and financial assumptions for the determination of reserves techniques shall be chosen on the basis of prudence, take account, where appropriate, of a margin reasonable for unfavourable variations; and are identified taking into account the following criteria:
        – the interest rates used in the calculation technical provisions shall be chosen on the basis of criteria of prudence, according to yield of the corresponding assets held by pension fund, expected returns of investments in a prudential scenario and taking into account the composition of the portfolio; [...]
        – the biometric tables used for the calculation technical provisions are based on principles prudential, in view of main characteristics of the members of the pension fund and changes anticipated in the relevant risks;
     (c) the method of assessment and the basis of calculation of technical provisions shall remain constant one financial year to another. Following changes in the legal, demographic or economic situation on which the assumptions are based may be made.
  4. Where the activities are not sufficient to cover technical provisions the pension fund is required to draw up a plan immediately of concrete and feasible rebalancing. [...]
  5. The recovery plan shall indicate, on the basis of concrete and achievable forecasts, the time required for the establishment of assets missing the complete coverage of reserves techniques.

6. When drawing up the plan, account shall be taken of: account of the specific situation of the pension fund and, in particular, the asset-liability structure, the associated risk profile, the liquidity needs, age profile of pensioners and active members. [...]

Art 5. (*Additional activities*):

1. Pension funds shall, on the basis of permanent, activities additional to the technical provisions referred to in art. 4 of this regulation. Such additional activities shall be designed to compensate for any difference between planned revenue and expenditure and actual over the period referred to in art. 3, paragraph 4 and are free from any predictable commitment.
2. The amount of the assets referred to in paragraph 1 shall be equal to 4% of the technical provisions of pension funds. [16]
3. The funds shall communicate to the COVIP the amount of assets referred to in paragraph 1 of this Article.

- Dlgs n. 252/2005 ex art. 7 - bis (*Assets*):

1. Pension funds which cover biometric risks, which guarantee a return on investment or a certain level of performance shall, in accordance with the criteria referred to in the following paragraph 2, equip themselves, appropriate capital resources in relation to all existing financial commitments, unless such financial commitments are undertaken by managers already subject to prudential supervision to which they are entitled, who operate in accordance with the rules governing them.
2. By regulation of the Ministry of Economy and Finance, after consulting COVIP, the Bank of Italy and ISVAP, the principles for the determination of adequate assets in accordance with the provisions of the Community provisions [...]
   2-bis. If the pension funds referred to in paragraph 1 that proceed with the direct payment of annuities do not have adequate assets in relation to all the existing financial commitments, the founding sources may redefine the discipline, in addition to the financing, benefits, both in respect of current and future annuities. Such determinations are sent to the COVIP for the assessments of competence. The possibility remains that the Funds' legal systems will give the internal bodies specific powers to restore balance to management.
3. The COVIP may, in respect of the forms referred to in paragraph 1, limit or prohibit the availability of assets where the appropriate assets have not been established in accordance with the regulation referred to in paragraph 2. The powers of the supervisory authorities over the managing entities remain unaffected.
   3-bis. The determinations referred to in paragraph 2-bis consider the

[16] a different percentage may be established in the cases described by paragraph 3 of this Article

objective of having a fair distribution of risks and benefits between generations.

- Covip - Circular letter on "the use of rating agencies' ratings from part of supplementary pension schemes" - 22/7/2013.
  The Funds' attention was pointed to the importance of specialized agency ratings being only one of the variables relevant to assessing the creditworthiness of debt securities issuers. Therefore other available information, where relevant, should not be excluded. Indeed, COVIP recognizes the necessity to take adequate measures to limit the use of rating judgments in investment and divestment decisions exclusively or mechanistically. Supplementary pension plans must include organizational processes and systems that guarantee an adequate creditworthiness assessment.
  Pension funds will therefore be careful to specify what is contained in the management agreements to date, so the person in charge of resource management uses appropriate credit assessment processes for debt securities issuers.

1.4.1 Asset Liability Management (ALM)

As the name suggests, Asset and liability Management is a key tool for establishing a link between assets and liabilities to manage interest rate risk, ensuring maximum profitability and profit stability ALM is an approach used by financial institutions to reduce financial risks caused by asset and liability mismatches. ALM techniques combine risk management with financial planning, and they're frequently employed by businesses to manage long-term risks that occur as a result of changing conditions. Financial institutions are left with a surplus after correctly matching assets and liabilities, which can be actively managed to enhance investment returns and increase profitability. Typically, only constant portfolio techniques are examined, such as investing a set amount in various sectors that remains constant throughout time (e.g. 30% of the capital into shares, 60% into bonds and the remaining 10% into real estate). It's worth noting that, at least roughly, we'd have to trade quite frequently, theoretically even at each time instant, to achieve this. The payout are determined based on an examination of the technical provisions, i.e. the best investment plan is determined by the evolution of the assets and liabilities. As a result, we must impose a constraint based on the investment strategy's riskiness, taking into account the asset value distribution and technical provisions. This can be done either to reduce the risk of default or to apply a risk measure.
Asset Liability Management is a set of methodologies and processes to support management choices in an integrated management of pension fund assets and liabilities. The objective is to optimize the trade off between expected return and risk assumed based on the available information and on the assumed future scenarios. ALM is not to be regarded as an immunization technique against market fluctuations, but rather as an instrument of decision and monitoring of management policies aimed at achieving adequate returns in relation to commitments and risks incurred.
It is clear that the model provided cannot ignore the study of the members' characteristics, the estimate of relative degree of tolerance to risk and the assessment of social security needs of the members. The statistical analysis of the demographic

members' composition and their distribution by age group and contributory seniority, by gender, educational qualification and salary levels is fundamental. The most representative groups are identified and within them the "typical" people with their demographic and economic-financial traits are constructed. In relation to social security needs, the projection of the participants, according to demographic characteristics, and the economic and financial variables must be carried out. In such a way as to calculate the corresponding basic pension and assess the replacement rates of the pension. In addition, the risk of various types of individuals in terms of predicted ultimate benefits must be calculated, and the contribution rate to the pension fund must be verified.
A change in the dynamics, consistency and/or riskiness of the expected flows of assets and liabilities (such as fluctuating interest rates or liquidity requirements) are typically the cause of mismatches. These changes in the financial landscape produces variations in value and creates the need to continuously manage flows (asset liability management) in order to avoid that activities in the medium to long term are not sufficient to face up to liabilities. By preserving liquidity needs, monitoring credit quality, and assuring enough operational capital, an ALM framework focuses on long-term stability and profitability. ALM, unlike other risk management techniques, is a well-coordinated process that use frameworks to monitor an organization's whole balance sheet. It guarantees that funds are invested as efficiently as possible and that liabilities are minimized throughout time.
The operator aims to maximize the surplus, which is the difference between value of assets (A) and liabilities (L) at each instant t ($S_t = A_t - L_t$).

The following factors must be considered while implementing a management model:

- Estimated expected flows from assets and liabilities
- Estimated discount rate of flows, consistent with risk
- Determination of the reference time horizon (both on assets and on liabilities) and calculation of duration
- Definition of a risk control measure
- Minimizing risk and maximizing yield with adequate diversification policies

Let's see schematically the approach just described.

Representation of an ALM model based on cash-flow matching approach

An ALM model is useful for asset management if it uses a cash flow matching method, which means it risks estimating the temporal distribution of assets and liabilities based on the differences between them. We list the components of formal representation of constrained optimal problem according to a cash-flow matching approach:

- $\{\tilde{x}(\alpha),\ \tilde{y},\ t\}$: vectors with random components of assets and liabilities on schedule t.
- $\tilde{x}(\alpha) = \{x_1, x_2, ..., x_m\}$: contributions and incomes of the pension scheme
- $\tilde{y} = \{y_1, y_2, ..., y_m\}$: social security benefits
- $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_p\}$: shares in composition of the portfolio consisting of p asset class

Assets $\tilde{x}(\alpha)$ depends on the composition shares of the p asset classes that make up the portfolio. In the generic deadline j, x_j post is linear combination of income deriving from the individual constituents of the asset classes (I_s for s-th constituent):

$$x_j = \sum_{s=1}^{p} \alpha_s I_s$$

The identification of optimal portfolio shares is therefore a problem of optimization (maximization) of an objective function f with respect to the quota vector α:

$$\max_{\alpha}\{f(\alpha_1, \alpha_2, ..., \alpha_p)\}$$

The objective function has as reference either a solvency indicator (such as the funding ratio) or an adequacy indicator (such as the replacement ratio).
The definition of *solvency constraint* in the optimization problem:

- $\tilde{\Delta}_k = x_k - y_k, \quad k = 1, 2, ..., m$ the difference between assets and liabilities on each deadline k
- Solvency condition of the pension scheme at the time of valuation t:

  $$\mathbb{E}_0\left[\sum_k \tilde{\Delta}_k \tilde{v}(t, t_k)\right] \geq 0$$

  where $\tilde{v}(t, t_k)$ is the discount factor.
  Therefore, the solvency requirement is based exclusively on the initial moment of the probability distribution of the random variable *present value of the balances* at time of evaluation, i.e. in 0. This statement expresses the fact that the discounted one-period balances' expected value must be strictly positive. This problem can be made more difficult by adding requirements that are higher in order than the first.

Technical balances depend on composition quotas, thus the problem of optimum shall return the set of α allowances which maximize the target function,but must also be such as to comply with solvency condition.
The definition of the target function must respect principles of adequacy of performance (replacement ratio target) and (financial) sustainability Solvency ratio or funding ratio (assets/value liability) target.

The model just illustrated is unquestionably consistent with the terms of the Ministry of Economy's Legislative Decree previously presented, as well as the identification of the defined objectives by the COVIP resolution of March 16, 2013.

Asset and liability management can encompass a variety of things, such as strategic asset allocation, risk reduction, and regulatory and capital framework adjustments.

Asset allocation

In terms of the COVIP resolution's requirements, we describe financial strategy as a method or criterion for allocating assets across a possible universe of investment opportunities in accordance with profitability aims while taking into consideration acceptable risk measures. It also tells verify the predicted level of the replacement ratio that the program is likely to attain, taking into consideration socio-demographic assumptions. As a result, we advise that we aim for the highest yield with the lowest risk, taking into account both assets and obligations.
To perform the asset allocation analysis must be determined the reference time horizon, the metric for assessing both the objective and riskiness, the classes of financial assets to be evaluated, investment limitations, and so on. Asset allocation refers to the practice of allocating capital among different investment classes in order to generate a diversified portfolio that is as responsive as possible to the needs of the investor (member of the pension fund) in terms of risk-return trade-offs.
In relation to the positions of interested parties, strategic goals that result in operational goals in terms of risk-return are specified. Identifying the investment classes on which to develop an effective portfolio will be crucial during the investment decision process (asset allocation). The performance is measured and evaluated. One of the most important parameters in the investment management evaluation process is the latter. In fact, performance allows members to voice their opinions on the manager's performance and, as a result, build penalty and incentive mechanisms in the member's best interests.

Let us now define a strategic control variable to which both the management bodies of the pension fund and the managers must refer for the asset allocation process.

Benchmark (BMK) The benchmark (BMK) is an index or combination of financial indices used to set the portfolio management technique, in compliance with the investment policy of the individual pension fund sector. Each linear combination coefficient is the weight with which the asset class is required to participate in the financial portfolio to cover the fund's commitments. A distinct benchmark is allocated to each segment into which assets are divided, which becomes the strategic variable utilized in the agreement with the financial manager to describe how it should move from an investing perspective. The benchmark not only has strategic importance (ex ante) in terms of management mandate entrustment, but also plays a strategic ex post role in terms of operator performance monitoring. Let's see the mathematical relationship of the linear combination of a set of stock market indices I_k *with* $k = 1, ..., N$, where α_k represent the respective shares:

$$BMK = \sum_{k=1}^{N} \alpha_k I_k, \qquad with \;\; 0 < \alpha_k < 1 \;\; and \;\; \sum_{k=1}^{N} \alpha_k = 1$$

Asset class k are related to each other differently, this produces mitigation effects and risk diversification.
In summary, the BMK has two functions:

- Defines the strategic asset allocation set by the Board of Directors
- Parameter against which the operator's efficiency is measured

The benchmark shall be *consistent* with the investment policy adopted and shall be *representative* of investment opportunities present on the market. It also requires the requirement of *objectivity*, that is, its components must be of common use, calculated and disseminated by subjects of undoubted reputation. And *transparency* is needed: in the sense that calculation must be easy and the rules of construction must therefore be known.

We see in more detail the functions of the benchmark both from the point of view of the adherent and that of the manager.
For the member, the BMK summarizes the risk-return profile of the investment, determining the typical risk of the market in which the manager invests. Thanks to this you can make choices more aware and more consistent with your profile risk-performance and objectively measure the quality of management (in the sense of ability of the operator). The member may also compare the product with related financial instruments. On the operator's side, the BMK identifies its task, namely to optimize the investment compared to the BMK. It allows to define in quantitative terms the objectives of the management and to continuously monitor the quality of the management, allowing a timely correction of errors. The operator can therefore conduct the asset allocation policy effectively and efficiently, strengthening the fiduciary relationship between the investor and the manager. In the case of direct management, the pension fund provides guidance on its marketing policies.

Financial strategies are categorized into passive or active management styles in comparison to the benchmark.

Strategic asset allocation A management style is characterized as passive (strategic asset allocation) when the manager manages the assets passively following the benchmark's composition in the execution of the mandate. Each of the asset classes defined in the linear combination is immediately replicated in the management operational investments in terms of quantity and quality of assets. The manager has at most the possibility to vary the choice of individual securities, provided they are part of the BMK. As a result, if the passive manager is efficient, it must always have a return and period volatility that are consistent with the yield and volatility of the benchmark to which it has been assigned. There are no over-performance criteria. The two typical moments of classical portfolio theory, such as average and variance, are used to assess the manager's effectiveness. A medium to long time horizon is required for such a strategy.

Tactical asset allocation The style of management is defined active (tactical Asset allocation), instead, the manager makes decisions about the selection of individual stocks (stock picking) and the moment in which to enter and exit from a given market (market timing). The management style is dynamic, the manager has the ability to vary weights relating to each asset class with respect to asset allocation indications strategic. The management mandate must provide a maximum risk level that the operator must adhere to when making dynamic decisions. Operationally speaking, namely that may overweigh or underweigh the asset classes identified by the benchmark due to decisions that are compatible with market expectations, but in accordance with the assumption of a risk (badget) imposed directly by the pension fund. Within the risk badget's limitations, the asset, unlike liabilities, may have higher volatility. Financial risk is higher in the portfolio than in the BMK. As a result, in comparison to the BMK, it is possible to attain higher performance. As a result, an over-performance fee will be charged to compensate the manager for being able to outperform BMK in terms of returns. The time horizon that is being evaluated right now is rather short.

The ALM models are currently divided into two groups.

Static models

The integrated asset and liability management is done in a single-period perspective, so the shares of the portfolio composition are not time-dependent. The optimal composition shares for each asset class are provided as a solution by the optimization problem, and they remain that way for the remaining life of both the financial portfolio and the social security commitments. These approaches have the advantage of being simple to use, but they produce a "one-dimensional" view, as if they were taking a "snapshot" of the assets and liabilities scenario. Changes in interest rates may have unintended consequences, which are not taken into account. The following are some examples of such models:

- Cashflow calendar
- Gap analysis
- Segmentation
- Cashflow matching

Dynamic models

As a result of the limited optimal issue, we now have an optimal set of composition quotas that are time dependent. Therefore , identifying a temporal guideline for recalibration of composition quotas over time is required. These models' management

strategies are based on simulations of a variety of scenarios (some of which are rather complicated) in order to provide guidance on the goal to be pursued. The preservation of a constant surplus; protection through a guaranteed minimum yield over a defined period of time; or, once again, the achievement of a certain degree of profitability are examples of typical aims.
According to the recalibration rule, there are two additional subcategories:

- Return driven model: The allowance recalibration rule is time-linked to the decision maker's yield target period.
- Value driven model : the allowance recalibration rule is based on the value trajectory of the individual asset classes that make up the portfolio. This typology can be divided into models that use passive strategies and models that use active strategies (in the latter subclass are placed the CPPI strategies that will be the main subject of this thesis)

The liability-driven strategy, which we shall explore briefly below has grown increasingly prominent among the static ALM models.

Liability-driven strategy

Liability Driven is certainly one of the most effective strategies to ensure acceptable levels of social security benefits from the point of view of social security coverage. This as they are based on the principle of perfect matching for maturity between assets-liabilities. A pension fund is an institutional investor with a medium-long time horizon term. In addition, it is able to determine precisely the time distribution of prospective social security commitments. The composition quota rule is based on the assumption that benchmark is not financial, but is represented the linear combination of passive posts.
We are therefore talking about a minimum risk strategy consisting of a combination of two portfolio strategies:

- *Liability Hedging Portfolio* (LHP): in an LDS it is the replicative portfolio of the benefits of the pension scheme. The time distribution of the LHP's cash flows is consistent with the time distribution of the expected commitments to be borne by the pension fund. LHP replicates liabilities through an appropriate linear combination of ZCB on different maturities and SWAP rates of interest. In particular, the proportion of ZCB's membership must be structured accordingly of the pension payments. The composition and weights of the LHP are determined on the basis of the risk aversion and the performance objectives of the pension fund.
- *Performance Seeking Portfolio* (PSP): is constituted for the maximization of invested patrimony in the time. The share invested in the PSP is positively correlated with the risk appetite, so as the risk appetite increases, the share invested in the PSP increases.

The portion investable in risky assets that is used to produce value on assets is the difference between the value of available assets and the value of LHP (Performance Seeking Portfolio). The difference between amounts of contributions (MC) valued at the valuation date and the present value of a portfolio of ZCB with various maturities is defined as *Free Capital*.

$$free\ capital = MC - N_T v(0,T)$$

- $v(0,T)$ = spot price of a unit ZCB
- N_T = certain equivalent of the social security benefit at maturity T

The portfolio's value at the initial time of the valuations is calculated as follows:

$$W_P(0) = \underbrace{N_T v(0,T)}_{LHP} + \underbrace{[MC - N_T v(0,T)]}_{free\ capital}$$

If it is invested in shares, multiply and divide the second member by $S(0)$, the share's original unit price, to get:

Where $\left[\frac{MC - N_T v(0,T)}{S(0)}\right]$ represents the share of the equity component (risky) and N the share of the non-risky portfolio.

Let's now look at another type of approach, which can be considered as a variation of the Liability Driven strategy.

Life Cycle Strategies

It is a strategy that automatically provides for the transition from riskier investment lines to more conservative ones over time. The aim is to reduce the risk of approaching the member of pension fund at time of entitlement to the benefits accrued. There is an automatic switching procedure between risky and no-risky investments to reach a predetermined age. As the age of adherent changes also its financial needs and in particular its propensity to risk. As a result, the investor's age plays a role in establishing the investment horizon. The strategy tries to produce personalized changes in investment over time. The basic requirement for implementation of the strategy is that the structure of pension fund can be ordered by compartments according to level of risk In this way it is possible to establish predefined paths between the different compartments.
Unlike the Liability Driven Investment plan, which focuses on the entire sector, the life cycle strategy tries to accomplish changes in the investment line that are customizable based on the scheme member's age.

Among the non-dynamic strategies, we distinguish the *Buy&Hold* strategies, through which, fixed a given benchmark, the composition shares of the asset classes are fixed and immutable over time and the *Constant Mix* strategies, by which the rule does not fix the absolute quotas of composition the portfolio, but the relative shares, that is to say, as time changes, if prices change the composition of the asset class in terms of relative weights. The dynamic strategy we highlight the Constant

Proportion, ie strategies which provide for a rule for the allocation of composition quotas depending on the time and behaviour of the financial market (there is a recalibration between risky and non-risky asset classes). Such strategists include the *Option Best Portfolio Insurance* (OBPI) strategy and the *Constant Proportional Portfolio Insurance* (CPPI) strategy. The latter will be dealt with in detail, and attention will be paid in the next chapter to the use of this investment strategy.

The strategies just named are part of that category of portfolio management techniques called: *Portfolio Insurance*.
In the following chapter we will introduce what portfoglio insurance means and describe Buy&hold, Constan Mix strategies and go into detail for CPPI. The latter will then be applied to an individual pension fund which is based on a defined contribution scheme.

Chapter 2

Portfolio Insurance

Portfolio insurance is a concept introduced by Mark Rubinstein and Hayne Leland in 1976 [12] to describe portfolio management approaches that aim to limit a portfolio's losses when stocks fall in value without forcing the portfolio manager to sell such stocks. Portfolio insurance techniques are intended to reduce downside risk while also profiting from rising markets. The goal of this technique is to ensure that the portfolio value at maturity or up to maturity is greater or equal to a certain lower bound (floor), which is commonly set as a percentage of the initial investment [H.E. Leland and M. Rubinstein, 1988 [13]].
Brennan and Schwartz were the first to notice the link between portfolio insurance and investment strategies, pointing out that insurance companies that had guaranteed the minimum payments they would make under equity-linked life insurance policies could hedge the resulting liability by using an investment strategy based on the Black-Scholes option-pricing model. The grouping of separate risks, or co-insurance, is the core idea of traditional insurance. Investors can minimize risk in a portfolio by using a similar approach on an equity portfolio, although the risk is only partially mitigated. Diversification alone is insufficient to protect portfolios, as investors will still be exposed to risks even if they diversify their investments to the utmost extent possible. This is due to massive variations in the stock market's overall trend, which are connected with equity returns. These risks can be further reduced by investing a larger percentage of one's portfolio in assets that are safe or risk-free, as well as by using financial strategies (such as portfolio insurance strategies) that implement dynamic portfolio rebalancing, and finally by replicating derivative instruments.
By balancing risk and projected return, portfolio insurance is an asset allocation or hedging approach that allows an investor to determine the level of risk he or she is prepared to accept. It is based on dynamic management, which is characterized by a small set of pre-established trading rules that regulate portfolio changes over time. The manager is asked to pre-determine the trading rules and the frequency with which the portfolio is rebalanced.

This chapter will focus on presenting Constant Proportion strategies, a particular type of Portfolio Insurance strategy. We will outline the market assumptions that we assume are valid for the remainder of the paper (unless otherwise indicated);

thereafter, we will present the basic elements for the implementation of such strategies. We will utilize the same method and notation as in Castellani G., De Felice M., Moriconi F. (2004) [17].

2.1 Market assumption

We present the main market assumptions that will guide the remainder of the article. We'll point out when this assumptions are not valid.

Arbitrage-free market For financial management we rely on the principle of no arbitrage. Arbitrage occurs when an investment strategy allows me, starting from zero capital, to have a certain profit in the future or to have a certain immediate profit without future commitments. We consider a market without arbitrage; in reality, the more a market is developed, the more it will conform to our assumption.

Hedging Portfolio This initial assumption is crucial in order to be able to use the replicating portfolio method. In order to be able to price complex contracts of which we do not know the value we can build portfolios formed by elementary contracts that replicate the cash flows of the complex contract. Because they are traded on a market, elementary contracts have a known value. The contract's cost must be identical to the cost of its replicating portfolio due to the premise of absence of arbitrage.

Complete market We consider a complete market. A market is complete if all securities can be replicated.When the number of sources of randomness exceeds the quantity of assets available, the market becomes incomplete.

2.2 Basic components of a Portfolio Insurance strategy

Consider an investment fund, with a market value F_t at time t.
The fund is divided into two components:

- a *managed fund*, consisting of an N_t^S number of *active asset*, with market value S_t; the value of the managed fund is:

$$E_t = N_t^S S_t; \tag{2.1}$$

  E_t is also called *exposure*;

- a *reserve fund*, consisting of an N_t^R number of *reserve assets*, with market value R_t; the value of the reserve fund is:

$$D_t = N_t^R R_t; \tag{2.2}$$

So we have:

$$F_t = E_t + D_t = N_t^S S_t + N_t^R R_t \tag{2.3}$$

The reserve component D_t is a non-risky fund, in the sense that replicates a fixed profile of liabilities with a good approximation; it can be understood as a "dedicated bond fund".
The managed component E_t is a risky fund, in the sense that it has unpredictable future value. You can think of it as wallet equity; it has a higher expected return than that of the reserve fund. In practical applications, the reserve fund is a portfolio of negligible risk compared to that of the managed fund.

Also consider:

- a fund with value B_t; the level B_t represents the value in t of the minimum guarantee (floor or bond floor). So, in the reference time frame, to avoid the shortfall we must have:

  $$F_t \geq B_t \qquad \forall t$$

  Ideally, the B_t value of the bond floor is non-risky (deterministic). In practical applications, B_t is the market value of a fund with very low risk. Often B_t has a comparable risk to D_t.

- a quantity C_t called *"cushion"*; the difference between the current portfolio value and the guaranteed amount:

  $$C_t = F_t - B_t \tag{2.4}$$

- a *self-financing* management strategy, described, as time changes, from the couple:

  $$(N_t^S, N_t^R)$$

  quotes N_t^S and N_t^R are readjusted according to a fixed rule.

A strategy is self-financing when the portfolio's F_t value resulting is fully reinvested, with no further payments either withdrawals, in the managed fund and in the reserve fund. That is, there must be:

$$\begin{aligned} F_{t+\Delta t} &= N_t^S S_{t+\Delta t} + N_t^R R_{t+\Delta t} \\ &= N_{t+\Delta t}^S S_{t+\Delta t} + N_{t+\Delta t}^R R_{t+\Delta t} \end{aligned} \tag{2.5}$$

Since:

$$N_t^S = \frac{E_t}{S_t} \qquad N_t^R = \frac{F_t - E_t}{R_t} \tag{2.6}$$

a management strategy can be set by defining the rules of calculation of the exposure E_t.

We can also define the *shareholding* in terms of the number of shares:

$$\alpha_t^N = \frac{N_t^S}{N_t^S + N_t^R} \tag{2.7}$$

or in terms of value

$$\alpha_t^V = \frac{E_t}{F_t} \tag{2.8}$$

2.3 Buy and Hold

The Buy and Hold ($B\&H$) strategy is defined by an initial mix of risky and non-risky components that remains constant throughout the investing period. So, this strategy does not need portfolio rebalancing, obviating the need for regular monitoring. As a result, this approach is known as a *"do-nothing"* strategy. As a result, the management expenses of a buy-and-hold strategy are modest.

The readjusting rule is therefore:

$$N_t^S = N_0^S, \quad N_t^R = N_0^R \qquad t > 0$$

The value of the fund for a generic instant t is equal to:

$$F_t = N_0^S S_t + N_0^R R_t$$

The equity share is constant in terms of shares:

$$\alpha_t^N = \alpha_0^N$$

but it varies in value:

$$\alpha_t^V = \frac{1}{1 + \frac{N_0^R}{N_0^S}\frac{R_t}{S_t}}$$

We can also write the background equation in the following way

$$F_t = \alpha_0^N S_t + (1 - \alpha_0^N) R_t$$

Remark 1. Suppose deterministic evolution for R_t:

$$R_t = R_0 e^{rt}$$

Furthermore, assuming $r = 0$ we have the *payoff function* of the type $y = ax + b$

$$F_t = \alpha_0^N S_t + (1 - \alpha_0^N) R_0$$

and the *exposure function* of the type $y = x - b$:

$$E_t = F_t - (1 - \alpha_0^N) R_0$$

With the bond exposition, this type of approach provides a safety net in the event of an equity market catastrophe. The deterministic level:

$$D_t = (1 - \alpha_0^N) R_t$$

forms a floor for the fund's F_t value. At the worst, the portfolio will perform as well as the floor. That is, its value will never fall below the initial risk-free asset investment.

Market movements change the risky and risk-free asset weightings. The value of a portfolio is proportional to the value of the stock market, that is, the proportion of shares (risky assets) in a buy-and-hold portfolio increases its relative value. The upward opportunities are potentially unlimited. To summarize, the buy-and-hold approach has the greatest potential for profit and loss. During a positive market, the higher the initial proportion invested in shares, the greater the payoff from a buy-and-hold approach. When the market is bearish, the opposite occurs. This method is, therefore, suitable for long-term investments of up to 15-20 years. Over extended periods of time, the equity market outperforms the bond market, and there are no multiple transaction costs.

2.4 Constant Mix

During the life of the investment, a Constant Mix (CM) approach keeps the total investment made on equity constant. The goal is to keep the strategic mix unchanged in the event of excessive market volatility, which would cause a portfolio rebalance. Every time the stock's value changes (in a different way from R_t), the manager must buy or sell the quotation in order to achieve the correct mix. If the stock's value rises, the management must sell shares or if the stock's value decline he must buy shares to allow the portfolio to be rebalanced. Constant Mix is called a *contravariant* strategy.

Therefore, in this case, the rule is to keep constant the equity share in terms of value:

$$\alpha_t^V = \alpha_0^V , \qquad t > 0$$

From (2.8) we derive the *exposure function*:

$$E_t = \alpha_0^V F_t$$

Investors that use Constant Mix strategies have risk tolerances that are proportional to their wealth. That is, there is a rule that requires an investor to buy and sell equities in response to changing market conditions in order to maintain a consistent portfolio composition. Investors have different rebalancing timings. Some portfolios are rebalanced at regular periods, although most are done when the portfolio's value has changed by a particular proportion. The Constant Mix strategy is under-performance while the stock market is rising, instead when the market is down sloping, the method also works.

2.5 Constant Proportion Portfolio Insurance

CPPI strategies are a particular type, a "Constant Proportion"(CP), of "portfolio insurance" (PI); they was originally studied by Black, Jones and Perold.

Constant Proportion strategy In a "Constant Proportion" (CP) strategy, a multiplier is defined:

$$m \geq 0$$

and the portfolio is managed in order to have:

$$E_t = m(F_t - B_t) = mC_t, \qquad t \geq 0$$

The portfolio is managed so that the equity exposure is always proportional to the cushion.

- A $B\&H$ strategy is a special case of a CP strategy (for $r = 0$). By placing:

  $$m = 1 \textit{ and } B_0 = D_0$$

  we have:

  $$E_t = F_t - D_0; \qquad t \geq 0$$

- A CM strategy is a special case of a CP strategy. By placing:

  $$m \in (0;1) \textit{ and } B_0 = 0$$

  we have

  $$E_t = mF_t$$

  that is:

  $$\alpha_t^V = m, \qquad t \geq 0$$

- CPPI is a CP strategy with multiplier greater than 1:

  $$m > 1$$

  We will now analyze this strategy in more depth.

The primary goal of portfolio insurance, as previously stated, is to give upside capture and downside protection.

CPPI The CPPI strategy is a self-financing approach (like other portfolio insurance techniques) that aims to leverage the returns of a risky asset through dynamic trading while guaranteeing a fixed amount of capital at maturity. To do so, the portfolio manager rebalances the risky and reserve asset components by keeping the portfolio risk exposure at a constant multiple of excess wealth on a floor, up to a borrowing limit. When a result, in order to accomplish upside capture, the number of owning shares must grow as the stock price rises. This is because, if the stock price rises, the payment will rise faster than the stock price if the holding shares rise; yet, if the holding shares remain unchanged, the payoff will rise proportionately to the stock price; and, worse, the payoff may fall if the holding shares fall. On the other hand, as the stock price falls, the number of shares must decrease in order to meet the purpose of downside protection. If the holding shares are decreasing, the payoff decreases more slowly than the stock price; however, if the holding shares remain unchanged, the payoff decreases proportionally to the stock price; and, worse, the payoff decreases more rapidly than the stock price if the holding shares are increasing.

The components described in the previous paragraph are considered.
The rule by which the shares N_t^S and N_t^R are adjusted is as follows:

$$E_t = N_t^S S_t = mC_t, \qquad \textit{where } m > 1 \ \textit{ is a fixed parameter} \tag{2.9}$$

The strategy assumes that the portion of value at risk (Exposure, E_t) is maintained (through periodic recalibration of the quotas of composition) equal to a prefixed multiple of the cushion.
Given that the exposure (2.9) constitutes the riskiest component of the investment portfolio F_t, the multiplier m defines the aggressiveness of the strategy: with the same cushion, larger m implies greater exposure.

If the stock market appreciates [depreciates] more than the bond floor, the cushion increases [decreases], and must therefore be increased [decreased] the number of shares. So, in a CPPI:

- a rise in S_t requires the purchase of shares
- a fall of S_t requires a sale of shares

CPPI is a *convex* strategy, which protects investors when a risky asset underperforms while failing to capture all of the gain when the risky asset recovers. As a result, convex methods are appropriate for portfolio insurance.
And furthermore CPPI is called a *tracking* strategy for the rebalancing mechanism. A CPPI performs well in the case of strongly rising markets; in the case of "fluctuating" markets, a CPPI can suffer from the same problems of a CM.

We note that even if m and B_0 are chosen so that $mC_0 \leq F_0$, it can happen that it results $mC_t > F_t$ for some $t > 0$; therefore the implementation of the strategy would require short selling the reserve fund (or a financing) for the amount:

$$mC_t - F_t$$

called financial leverage.
It is said that the *borrowing limit* is reached when $mC_t = F_t$, that is, when $F_t = B_t \frac{m}{m-1}$.

Typically there is an upper limit on leverage by imposing on the exposure a maximum admissible value equal to lF_t; with $1 \leq l < m$. The definition of the strategy therefore becomes:

$$E_t = \min\{mC_t; lF_t\}, \qquad 1 \leq l < m \tag{2.10}$$

the coefficient l is the so-called *maximum leverage ratio*.
The goal of the CPPI is to ensure that it is $F_t \geq B_t$ for each t by safeguarding the

chances of earning in a bull market. The bond floor B_t it must be defined both in the level and in the composition. Once the characteristics of the managed fund have been specified, the strategies CPPI can still differ greatly in choice of the reserve fund and the shape of the floor.
If in an instant t^* it is $F_{t^*} = B_{t^*}$ (and therefore $E_{t^*} = 0$) for the first time, the strategy hit the bond floor and so for any $t > t^*$ we will have $F_t = D_t$.

In summary.
The investor begins by deciding on a minimum acceptable value for the portfolio as a guarantee level. She or he then calculates the cushion and multiplies it by a specified multiplier to determine the amount to be allocated to the risky asset. The model's exogenous inputs are the floor and multiplier, which indicate the risk tolerance of the investor. A greater multiplier number indicates a low risk aversion, allowing the investor to participate more in the stock market. This would allow you to profit from a market upswing. In the event of a persistent drop in stock prices, however, it would imply that the portfolio would reach the floor sooner.Because the exposure is a function of the cushion, it approaches zero when the cushion does. This ensures that the portfolio value does not fall below the floor level. Only a sudden dramatic decline in the market before the investor has an opportunity to trade will cause the portfolio value to go below the floor.
It's worth noting that the floor process B has to be hedgable, and the explicit form of the related hedging strategy has to be calculable. Otherwise, the portfolio insurance would fail since the corresponding allocation would be unknown.

2.5.1 Risk linked to CPPI

There are various risks connected with a CPPI approach that are not present in other investment schemes. The *cash-lock* risk and the *gap-risk*, both of which are detailed below, are the most prevalent. These risks will be examined in the following chapters under various market assumptions.

Cash-Lock Risk One of the most significant risks for a CPPI portfolio manager in a discrete-time trading market is the so-called cash-lock phenomenon, which refers to the consequence that, owing to losses, the lower bound for the guarantees can only be fulfilled via riskless investing from a certain point onward. It is the possibility of the portfolio being monetized. This is when all of the portfolio's assets are totally invested in a risk-free asset with no opportunity of recovering. Because a cash-locked position limits any participation in a rising market, it is seen as a critical risk for long-term investing. When the exposure reaches zero, it remains there until the investment term ends. This risk is less of a problem in continuous-time since dynamic trading is possible and investors may rapidly relocate money when the buffer approaches zero. As a result, this risk is mostly investigated in markets with discrete-time trading, which will be explored in this thesis. This problem can be avoided in a framework where floor interference is feasible by moving the floor downwards according to market conditions. In this sense, the CPPI approach avoids the so-called cash-lock risk by preventing the portfolio from being monetized.

Gap Risk Another significant risk that a portfolio manager confronts while managing a CPPI portfolio is gap risk. It's the probability of a portfolio's value crashing through the floor, or dropping below the floor, resulting in a negative gap. When a large portion of one's money is invested in equities, and the fund management does not have enough time to rebalance the portfolio in the event of a significant market downturn, this situation emerges. Because this might result in the plan's promised amount not being met at the end, it's critical to identify and hedge this "gap risk."

2.5.2 Standard CPPI in continuous time

We consider a CPPI strategy in a continuous-time setup with continuous asset paths, therefore the recalibration of the shares N_t^S and N_t^R occur at every instant, and the process S_t has continuous trajectories. In this case the strategy is effective, i.e. the inequality is guaranteed:

$$F_t \geq B_t, \qquad \forall t$$

Obviously, with continuous rebalances, profit will be eroded by transaction costs, but for the moment we do not consider them.

It is now necessary to define the evolution of the equity component and the bond component.
Let's consider: S_t the price in t of a unit of risky asset and R_t the price in t of a unit of riskless bond.
The reference model for the evolution of the riskless asset's value R_t is the exponential deterministic growth with constant risk-free rate r:

$$dR_t = rR_t dt \tag{2.11}$$

We can think of this asset as the value of money market account.
From (2.11) we have:

$$R_t = R_0 e^{rt} \tag{2.12}$$

Proof. With simple steps and considering the interval $[0, t]$ we can verify what was said previously.

$$\begin{aligned}
&\frac{dR_t}{R_t} = rdt \\
&d\ln R_t = rdt \\
&\int_0^t d\ln R_u \, du = \int_0^t r \, du \\
&\ln R_t - \ln R_0 = r(t-0) \\
&R_t = R_0 e^{rt}
\end{aligned}$$

□

Instead, for the equity component, we assume that S_t evolves stochastically; S_t follows a general diffusive process:

$$dS_t = \mu(t; S_t)dt + \sigma(t; S_t)dW_t \tag{2.13}$$

In this expression $\mu(t; S_t)$ is the deterministic drift and $\sigma(t; S_t)$ is the volatility; W_t is a Brownian motion, i.e. a stochastic process such that all increments are independently and normally distributed with null expectation and variance equal to the time step and dW_t denotes the infinitesimal increments of the Brownian motion

$$dW_t = W_{dt+t} - W_t \sim N(0; dt)$$

Brownian motion W_t is defined on the full probability space $(\Omega; \mathbb{F}; \mathbb{P})$, which is filtered by $\{F_t\}_{t\in[0;T]}$. (For more detailed information see A.1)
Precisely it is assumed that S_t is a geometric Brownian motion, described by the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.14}$$

That is *Black–Scholes model.* The parameters σ and μ are real constants with $\sigma > 0$. Thanks to this assumption we are able to define the distribution of S_t:

$$S_t \sim LogN(\mu; \sigma^2)$$

We can solve the differential equation (2.14) applying the itô's lemma (see A.2), obtaining:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma dW_t} \tag{2.15}$$

One can write $dW_t = W_t - W_0 = \sqrt{t}\ \epsilon_t$ where $\epsilon_t \sim N(0, 1)$.
The expected value of the asset component S_t is

$$\mathbb{E}(S_t) = S_0 e^{\mu t} \tag{2.16}$$

and the instantaneous expected return

$$\mathbb{E}\left(\frac{dS_t}{S_t}\right) = \mu dt$$

We assume the guarantee of the CPPI portfolio is G. It is the least payment the investor should receive at maturity T. Set B_t to be the present value of G, discounted by the riskfree rate r, which forms the Floor. Consequently, the insurance issuer is obligated to cover the difference between the guarantee and the final portfolio value. Mathematically we describe the Floor in the following way.

$$B_t = Ge^{-r(T-t)} \tag{2.17}$$

$$dB_t = rB_t dt \tag{2.18}$$

The portfolio's floor B_t, which serves as a lower constraint, is expected to rise in accordance with the risk-free rate r.
The starting floor, B_0, is obviously lower than the initial portfolio value, F_0. The investor's goal, according to the approach, is to have a portfolio value of F_0 at retirement, which means she doesn't want any capital losses. Because a risk-free asset exists in the market in which he or she may invest the entire floor, one would only require an initial capital of $B_0 = F_0 e^{rT}$ at time $t = 0$ to fulfill this aim. As a result, the investor can take a risk with the extra amount $F_0 - B_0 = C_0$ (the cushion at time $t = 0$) without any probability of not meeting the final payoff.
It is clear that R_t and B_t $\forall t$ have the following relationship, based on their dynamics (2.13) and (2.18):

$$\frac{dR_t}{R_t} = \frac{dB_t}{B_t} \qquad \forall\ t$$

Let's consider

$$N_t^S = \frac{mC_t}{S_t} \qquad N_t^R = \frac{F_t - mC_t}{R_t} \tag{2.19}$$

The dynamics of the fund F_t can be written as follows

$$\begin{aligned} dF_t &= N_t^S dS_t + N_t^R dR_t \\ &= \frac{mC_t}{S_t}\Big[\mu S_t dt + \sigma S_t dW_t\Big] + \frac{F_t - mC_t}{R_t}\Big[rR_t dt\Big] \\ &= \Big[mC_t\mu + (F_t - mC_t)r\Big]dt + \Big[mC_t\sigma\Big]dW_t \\ &= \Big[mC_t(\mu - r) + F_t r\Big]dt + \Big[mC_t\sigma\Big]dW_t \end{aligned} \tag{2.20}$$

From which one can derive the cushion equation

$$\begin{aligned} dC_t &= dF_t - dB_t \\ &= \Big[mC_t(\mu - r) + F_t r\Big]dt + \Big[mC_t\sigma\Big]dW_t - rB_t dt \\ &= \Big[mC_t(\mu - r) + (F_t - B_t)r\Big]dt + \Big[mC_t\sigma\Big]dW_t \\ &= \Big[mC_t(\mu - r) + C_t r\Big]dt + \Big[mC_t\sigma\Big]dW_t \\ &= (r + m(\mu - r))C_t dt + m\sigma C_t dW_t \end{aligned} \tag{2.21}$$

It is clear to see that the cushion process C_t follows a geometric Brownian motion with mean and volatility denoted, respectively, by μ_c and σ_c

$$\mu_c = r + m(\mu - r)$$

$$\sigma_c = m\sigma$$

The cushion equation becomes

$$dC_t = \mu_c C_t dt + \sigma_c C_t dW_t \tag{2.22}$$

Consequentially

$$C_t = C_0 e^{\left(\mu_c - \frac{\sigma_c^2}{2}\right)t + \sigma_c W_t} \tag{2.23}$$

$$C_t \sim LogN(\mu_c; \sigma_c^2)$$

Knowing that

$$F_t = C_t + B_t = C_0 e^{\left(\mu_c - \frac{\sigma_c^2}{2}\right)t + \sigma_c W_t} + Ge^{-r(T-t)}$$

it follows therefore that the value of the investment portfolio is described by the sum of a deterministic component and a stochastic component. The deterministic component grows with intensity r. The stochastic component follows a lognormal process with instantaneous mean $\mu_c = r + m(\mu - r)$ and instantaneous volatility $\sigma_c = m\sigma$.

Let's now analyze the relationship between C_t and S_t.
Raising the (2.15) equation to the m-th power

$$\left(\frac{S_t}{S_0} e^{-\left(\mu - \frac{\sigma^2}{2}\right)t}\right)^m = \left(e^{\sigma\sqrt{t}\epsilon_t}\right)^m$$

And from (2.23)

$$\frac{C_t}{C_0} e^{-\left(\mu_c - \frac{\sigma_c^2}{2}\right)t} = e^{\sigma_c\sqrt{t}\epsilon_t}$$

Since the second terms are equal we can equal

$$\frac{C_t}{C_0} e^{-\left(\mu_c - \frac{\sigma_c^2}{2}\right)t} = \left(\frac{S_t}{S_0}\right)^m e^{-m\left(\mu - \frac{\sigma^2}{2}\right)t}$$

$$\begin{aligned} C_t &= C_0 \left(\frac{S_t}{S_0}\right)^m e^{\left[\left(\mu_c - \frac{\sigma_c^2}{2}\right) - m\left(\mu - \frac{\sigma^2}{2}\right)\right]t} \\ &= C_0 \left(\frac{S_t}{S_0}\right)^m e^{\left[r - m\left(\frac{r - \sigma^2}{2}\right) - \frac{m^2\sigma^2}{2}\right]t} \\ &= C_0 \left(\frac{S_t}{S_0}\right)^m e^{-(m-1)\left[r + m\frac{\sigma^2}{2}\right]t} \end{aligned}$$

From which we can clearly see that the value of the cushion C_t in a generic instant t is independent of the trajectory of the underlying, but it only depends from its current value S_t. Consequently, even the fund

$$F_t = Ge^{-r(T-t)} + C_0 \left(\frac{S_t}{S_0}\right)^m e^{-(m-1)\left[r + m\frac{\sigma^2}{2}\right]t}$$

does not depend on the trajectory of the asset, it is therefore said that the trend of F_t is *path independent*.
Even if the value of the underlying were to cancel out, it would however $F_t =$

$Ge^{-r(T-t)}$. The stochastic component of F_t is determined by appreciation of S_t, through the factor $\left(\frac{S_t}{S_0}\right)^m$, attenuated by the factor:

$$e^{-(m-1)\left[r+m\frac{\sigma^2}{2}\right]t}$$

Finally calculating the expected value and the variance (at time 0) of the portfolio value of investment:

$$\mathbb{E}_0(F_t) = B_t + E(C_t) = Ge^{-r(T-t)} + C_0e^{\mu_c t}$$

$$\mathbb{V}_0(F_t) = C_0^2 e^{2\mu_c t}(e^{\sigma_c^2 t} - 1)$$

we arrive to the paradoxical conclusion that in the Black–Scholes model, whenever $\mu > r$, the expected return of a CPPI portfolio $\mathbb{E}_0(F_t)$ can be increased indefinitely and without risk, by taking a high enough multiplier.

The fact that the strategy cannot fail is a consequence of assumptions made on the nature of the stochastic process for S_t. By choosing a continuous process and allowing the continuous recalibration "one can manages to track" the changes in the process, avoiding the downturns.
But the possibility of going below the floor, known as *gap risk*, is widely recognized: there is a nonzero probability that, during a sudden downside move, the fund manager will not have time to readjust the portfolio, which then crashes through the floor. In this case, the issuer has to refund the difference, at maturity, between the actual portfolio value and the guaranteed amount.
Therefore continuous rebalancing is not a reasonable assumption; CPPI must be carried out in a discrete time.

2.5.3 Standard CPPI in discrete time

Now we consider a CPPI carried out in a discrete time: readjustments odds are performed at finite time intervals (i.e. every day). Consider a standard CPPI. Let Δt be the time interval between two successive recalibrations and let $i = e^{r\Delta t} - 1$ the interest rate for the period $[t; t + \Delta t]$; since:

$$D_{t+\Delta t} = D_t(1 + i) = (F_t - mCt)(1 + i)$$

and

$$E_{t+\Delta t} = mCt\frac{S_{t+\Delta t}}{S_t}$$

we can derive the relation:

$$F_{t+\Delta t} = D_{t+\Delta t} + E_{t-\Delta t} = F_t(1 + i) + mC_t\left(\frac{S_{t+\Delta t} - S_t}{S_t} - i\right) \tag{2.24}$$

The steps of the strategy. The realization of a CPPI strategy with recalibration step Δt can be summarized in the following steps:

1. Calculation of the NAV
$$F_t = N^S_{t-\Delta t} S_t + N^R_{t-\Delta t} R_t$$
2. Calculation of the cushion
$$C_t = F_t - B_t$$
3. Calculation of the exposure
$$E_t = mC_t$$
4. Calculation of the composition quotas
$$N^S_t = \frac{E_t}{S_t}, \qquad N^R_t = \frac{F_t - E_t}{R_t}$$

Evolution of the cushion Consider a strategy with duration $T = n\Delta t$, that is, with n instants of recalibration, $t_k = k\Delta t; k = 1, 2, ...n$. If there is no shortfall up to t_k (or up to $t_k - 1$), from (2.29), for $k = 1, 2, ...n$, we have:

$$C_k = mC_{k-1}\left(\frac{S_k}{S_{k-1}} - \lambda_t\right)$$

from which

$$C_k = m^k C_0 \prod_{h=1}^{k} \left(\frac{S_h}{S_{h-1}} - \lambda_t\right)$$

The value of $F_k = F(t_k)$ is therefore given by:

- if there is no shortfall up to t_k:
$$F_k = C_k + B_k; k = 1, 2, ...n$$
- if the shortfall occurs in $0 < t_h \leq t_k$:
$$F_k = (C_h + B_h)e^{r(k-h)\Delta t}; h = 1, 2, ...k; k = 1, 2, ...n$$

Therefore the random variable $F(T) = F_n$ can be expressed as:

$$F_n = C_n + B_n \mathbf{1}_{\tau > T} + \sum_{h=1}^{n} (C_h + B_h)e^{r(k-h)\Delta t} \mathbf{1}_{\tau = t_h} \tag{2.25}$$

Having indicated with 1E the indicator function of the event E. It is easy to verify that:

$$\mathbf{1}_{\tau > T} = \prod_{k=1}^{n} \mathbf{1}_{\Gamma_k}, \qquad \mathbf{1}_{\tau = t_h} = \prod_{k=1}^{h-1} \mathbf{1}_{\Gamma_k}(1 - \mathbf{1}_{\Gamma_k})$$

being:

$$\Gamma_k = \left\{\frac{S_k}{S_{k-1}} > \lambda\right\}$$

Then:

$$\mathbb{P}(\tau > T) = \mathbb{E}_0[\mathbf{1}_{\tau > T}] = \mathbb{E}_0[\prod_{k=1}^{n} \mathbf{1}_{\Gamma_k}]$$

$$\mathbb{P}(\tau = t_h) = \mathbb{E}_0[\mathbf{1}_{\tau = t_h}] = \mathbb{E}_0[\prod_{k=1}^{h-1} \mathbf{1}_{\Gamma_k}(1 - \mathbf{1}_{\Gamma_k})]$$

Shortfall There is evidence of the gap risk mentioned at the beginning of the chapter in discrete trading. So, in this scenario, let's look at a measure for this risk: the probability of a shortfall. This is the probability that the fund's value will fall below the bond floor's value, given that the fund's value was previously greater than the bond floor's value[1].
Let $F_t > B_t$. If the price of active assets falls strong enough between t and $t + \Delta t$, it may happen that the strategy fails, that is, we have $F_{t+\Delta t} \leq B_{t+\Delta t}$ ("*shortfall*"). At time t, define the probability of failure, or shortfall, φ_t as the conditional probability:

$$\varphi_t = \mathbb{P}_t(F_{t+\Delta t} \leq B_{t+\Delta t}|F_t > B_t) \tag{2.26}$$

The probability φ_t can be expressed by introducing the "shortfall factor" λ_t, defined as the value of $\frac{S_{t+\Delta t}}{S_t}$ for which $F_{t+\Delta t} = B_{t+\Delta t}$.

$$\varphi_t = \mathbb{P}_t\Big(\frac{S_{t+\Delta t}}{S_t} \leq \lambda_t|F_t > B_t\Big) \tag{2.27}$$

By subtracting $B_{t+\Delta t} = B_t(1+i)$ from both sides of the relationship (2.24)

$$C_{t+\Delta t} = C_t\Big[(1+i) + m\Big(\frac{S_{t+\Delta t} - S_t}{S_t} - i\Big)\Big] \tag{2.28}$$

For $\frac{S_{t+\Delta t}}{S_t} = \lambda_t$ there must be $C_{t+\Delta t} = 0$, that is:

$$(1+i) + m[\lambda_t - (1+i)] = 0$$

from which we obtain:

$$\lambda_t = (1+i)\frac{m-1}{m} \tag{2.29}$$

The shortfall factor λ_t depends only on m and i, which however is a function increasing of Δt. Fixed the characteristics of S_t, the greater is Δt the more the probability that the strategy fails between t and $t + \Delta t$ is high. Furthermore, the more the strategy is aggressive (m large) the lower the shortfall threshold and the higher is the probability that the strategy will fail.

The hypothesis of geometric Brownian motion behind the standard model (2.14) is equivalent to assuming that:

$$\ln\frac{S_{t+\Delta t}}{S_t} \sim N\Big((\mu - \frac{\sigma^2}{2})\Delta t, \sigma\sqrt{\Delta t}\Big)$$

We therefore obtain:

$$\begin{aligned}\varphi_t &= \mathbb{P}_t\Big(\ln\frac{S_{t+\Delta t}}{S_t} \leq \ln\lambda_t\Big)\\ &= \mathbb{P}_t\Big(\frac{\ln\frac{S_{t+\Delta t}}{S_t} - (\mu - \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}} \leq \frac{\ln\lambda_t - (\mu - \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}\Big)\end{aligned}$$

where

$$\frac{\ln\frac{S_{t+\Delta t}}{S_t} - (\mu - \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}} \sim N(0,1)$$

[1] For accuracy it is the definition of *local* shortfall probability

Therefore

$$\varphi_t = \Phi\left(\frac{\ln \lambda_t - (\mu - \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}\right) \tag{2.30}$$

By replacing $\lambda_t = (1+i)\frac{m-1}{m} = \frac{m-1}{m}e^{r\Delta t}$ in (2.30) we clearly see the dependence of

$$\frac{\ln\left(\frac{m-1}{m}e^{r\Delta t}\right) - (\mu - \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}$$

from $m, r, \Delta t, \mu$ *and* σ.

Consider a strategy with constant t recalibration intervals. Given the characteristics of geometric Brownian motion, the events:

$$\frac{S_{k\Delta t}}{S_{(k-1)\Delta t}} \le \lambda$$

are independent and all have equal probability, given by:

$$\varphi_t = \Phi\left(\frac{\ln \lambda - (\mu - \frac{\sigma^2}{2})\Delta t}{\sigma\sqrt{\Delta t}}\right)$$

hence the probability of the shortfall occurring for the first time during the k-th period is given by:

$$p_k = (1-\varphi)^{k-1}\varphi \tag{2.31}$$

Hence, assuming $q = (1-\varphi)$, the probability that the shortfall will occur for $t \le k\Delta t$ is:

$$\mathbb{P}_0(\tau \le k\Delta t) = \sum_{h=1}^{k} p_h = \varphi \sum_{h=1}^{k} q^{h-1} = \frac{1-q}{q}\sum_{h=1}^{k} q^h$$

From which:

$$\mathbb{P}_0(\tau \le k\Delta t) = \frac{1-q}{q}\frac{q(1-q^k)}{1-q} = 1 - q^k = 1 - (1-\varphi)^k$$

Waiting time, average time The expected time (or average time) to shortfall can be defined as the expectation:

$$\bar{\tau} = \mathbb{E}_0(\tau)$$

of the random variable (waiting time):

$$\tau = \min\left\{t : \frac{S_t}{S_{t-\Delta t}} \le \lambda_t\right\}$$

For a strategy with constant Δt, we also have:

$$\tau = \nu\Delta t$$

where:

$$\nu = \min\left\{k : \frac{S_{k\Delta t}}{S_{(k-1)\Delta t}} \le \tau\right\}$$

is the number of recalibration periods up to (inclusive) shortfall.

Consider a strategy with duration $T = n\Delta t$, that is, with n periods at expiration. Remembering the (2.31). The probability of no shortfall occurring within the deadline is:

$$q_n = (1-\varphi)^n$$

the average value of ν is given by:

$$\begin{aligned}
\bar{\nu}_n = \mathbb{E}_0[\nu] &= \sum_{k=1}^{n} kp_k + nq_n \\
&= \frac{\varphi}{1-\varphi}\sum_{k=1}^{n} k(1-\varphi)^k + n(1-\varphi)^n \\
&\quad bacause\ of\ \sum_{k=1}^{n} k(1-\varphi)^k = \frac{1-\varphi}{\varphi}\left[\frac{1-(1-\varphi)^n}{\varphi} - n(1-\varphi)^n\right] \\
&= \frac{1-(1-\varphi)^n}{\varphi}
\end{aligned}$$

The expected shortfall time is therefore:

$$\bar{\tau}_n = \bar{\nu}_n\Delta t = \frac{1-(1-\varphi)^n}{\varphi}\Delta t$$

In particular, for a strategy with a potentially unlimited duration we have:

$$\bar{\nu}_\infty = \lim_{n\to\infty}\frac{1-(1-\varphi)^n}{\varphi} = \frac{1}{\varphi}$$

$$\bar{\tau}_\infty = \bar{v}_\infty\Delta t = \frac{\Delta t}{\varphi}$$

Of course, for large values od Δt the measure of expected time $\bar{\tau}$ it may be inaccurate, given the *granularity* error.

Chapter 3

CPPI in defined contribution pension plan

The uncertainty surrounding future benefits, as well as the fact that the pension fund participants is directly exposed to the plan portfolio's financial risk, make pension fund modeling crucial. In this environment, downside protection against market circumstances has become particularly important. Because the benefits of DC plans are not predetermined, the member is directly exposed to risk, providing a minimum guarantee for the fund is critical. This minimum guarantee sets a lower limit on the amount of money that will be paid out to members in retirement, providing downside protection against investment risk. Another issue that has emerged in portfolio protection/insurance is determining the amount of guarantee that will be paid after retirement. It is important to remember that the fund's worth is made up of the participant's contribution payments and the portfolio's return on these payments. As a result, it's also crucial to consider the stochasticity of the payments, which adds another layer of randomness to the benefit. To meet this demand, the primary goal of this study is to develop a protected portfolio plan for DC pension funds that accounts for the randomness resulting from contribution levels while also ensuring at least the cumulative level of wealth.
The current thesis is devoted to the analysis and construction of a set of insurance strategies in defined contribution (DC) type pension funds with the goal of creating the mathematical theory for these types of protection schemes.

We now go on to the core of the presented thesis, keeping in mind the preliminary concepts presented in the first and second chapters. We will proceed with the treatment suggested in Korn R., Selcuk-Kestel A. S., Temocin B. Z. (2017) [26] and Temocin B.Z. (2015) [38].
The fund management time horizon is considered to be $[0; T]$, where time $t = 0$ is the date of enrollment in the plan and T is the date of retirement. We take into account contributions that are made on a regular basis (i.e. monthly) and are calculated as a percentage of the contributor's work income. This income is also characterized as a stochastic process that reflects the risks of financial market. These CPPI techniques are variants of the traditional CPPI described in the previous chapter, and they are used to administer DC pension plans. The market environment, as well as the two

suggested floor processes, are explained in the following sections.

3.1 Defined contribution model

We should first impose some assumptions in order to model the flow of contributions. Assume that each beneficiary contributes a specified percentage of his or her work income, ζ, at predetermined periods $t \in [0;T]$, and that the pension plan involves a continuous flow of contribution payments. Because there are numerous unpredictable elements such as disability, death, and economic or political crises, realistic modeling of labor income can be challenging. We can suppose that labor income $I(t)$ is a stochastic process that satisfies the following stochastic differential equation:

$$\begin{cases} dI_t = \mu_I I_t dt + \sigma_I I_t dW_t \\ I_0 = l_0 \end{cases} \tag{3.1}$$

μ_i and σ_i are both considered to be real constants, W_t is a Brownian motion defined as in previous chapter. We will use either $I(t)$ or I_t notation to indicate the process at time t.
We recall the equations (2.14) and (2.15) related to the risky component of CPPI strategy outlined above. As a result, the labor income at time t is given as

$$I_t = I_0 e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)t + \sigma_I dW_t} \tag{3.2}$$

Now imagine dividing the time interval $[0;T]$ into n instants from $t_0 = 0$ to $t_n = T$. Following this approach, we can say that every contribution $\zeta(t_i)$ $\forall i = 0,1,2,\dots,n$ has the form

$$\zeta(t_i) = \zeta I(t_i) \qquad \forall i = 0,1,2,\dots,n \tag{3.3}$$

with the dynamics

$$d\zeta(t) = \zeta dI(t) \tag{3.4}$$

given that $t \in (t_i, t_{i+1})$ for $i = 0,1,2,\dots,n$.

All of the processes discussed thus far have been described using a *real-word probability*, denoted by $\mathbf{P}$. However, in order to analyze the request for future donations, we must use a different measure of probability. The risk-neutral measure (see Appendix B) that we will express with $\mathbf{Q}$, which includes the *risk premium* (ore th *market price of risk*), is the measure to which we will refer for this purpose. We can say that the risk-neutral measure exists and is unique because of the market assumptions made above; in particular, for diffusion process, we can say that the no-arbitrage principle ensures that measure $\mathbf{Q}$ exists, and the market completeness ensures that this measure is unique (B.1). In summary, A unique equivalent martingale measure $\mathbf{Q}$ exists in a full market, under which the unique price of a contingent claim may be estimated as an expectation. Using Girsanov's (B.1) theorem as a main basis, we construct the risk-neutral dynamic of $I(t)$, which is given by

$$dI_t = (\mu_I - \sigma_I \lambda) I_t dt + \sigma_I I_t d\widetilde{W}_t \tag{3.5}$$

where

$$\lambda = \frac{\mu - r}{\sigma} \tag{3.6}$$

is the market price of risk and $\tilde{W}_t$ is the Brownian motion under $\mathbf{Q}$.

$\zeta(t)$ is an F_t-measurable random variable[1] and $\mathbf{Q}$ is a martingale measure on the market for the underlying. As a result it is possible to apply *the fundamental pricing rule* for determine the time-t price of the stream of future contributions payable between t and T. Let $\Lambda(t)$ denote the price which is defined as the summation of discounted future payments and is given by

$$\Lambda(t) = \mathbb{E}_t^{\mathbf{Q}}\left[\sum_{i:t_i \geq t} e^{-r(t_i-t)}\zeta(t_i)\middle| F_t\right] \tag{3.7}$$

By applying Itô formula, the stochastic differential equation (3.5) has a solution given by :

$$I(t_i) = I(t)e^{\left(\mu_I - \sigma_I\lambda - \frac{\sigma_I^2}{2}\right)(t_i-t)+\sigma_I\left(\widetilde{W}(t_i)-\widetilde{W}(t)\right)} \qquad \forall t_i > t \tag{3.8}$$

Thanks to the aforementioned result, we can explain $\Lambda(t)$:

$$\begin{aligned}\Lambda(t) &= \mathbb{E}_t^{\mathbf{Q}}\left[\sum_{i:t_i \geq t} e^{-r(t_i-t)}\zeta I(t)e^{\left(\mu_I - \sigma_I\lambda - \frac{\sigma_I^2}{2}\right)(t_i-t)+\sigma_I\left(\widetilde{W}(t_i)-\widetilde{W}(t)\right)}\middle| F_t\right] \\ &= \zeta\mathbb{E}_t^{\mathbf{Q}}\left[\sum_{i:t_i \geq t} e^{-r(t_i-t)}I(t)e^{\left(\mu_I - \sigma_I\lambda - \frac{\sigma_I^2}{2}\right)(t_i-t)+\sigma_I\left(\widetilde{W}(t_i)-\widetilde{W}(t)\right)}\middle| F_t\right]\end{aligned} \tag{3.9}$$

Since $I(t)$ is F_t-measurable, it follows that

$$\begin{aligned}\Lambda(t) &= \zeta\mathbb{E}_t^{\mathbf{Q}}\left[\sum_{i:t_i \geq t} e^{-r(t_i-t)}I(t)e^{\left(\mu_I - \sigma_I\lambda - \frac{\sigma_I^2}{2}\right)(t_i-t)+\sigma_I\left(\widetilde{W}(t_i)-\widetilde{W}(t)\right)}\right] \\ &= \zeta\sum_{i:t_i \geq t} e^{-r(t_i-t)}I(t)e^{\left(\mu_I - \sigma_I\lambda - \frac{\sigma_I^2}{2}\right)(t_i-t)}\mathbb{E}_t^{\mathbf{Q}}\left[e^{\sigma_I\left(\widetilde{W}(t_i)-\widetilde{W}(t)\right)}\right]\end{aligned} \tag{3.10}$$

Knowing that $\widetilde{W}(t_i) - \widetilde{W}(t) \sim N(0, t_i - t)$ we can recognize that the last expectation is the moment generating function of the normal distribution written above. Therefore:

$$E_t^{\mathbf{Q}}\left[e^{\sigma_I\left(\widetilde{W}(t_i)-\widetilde{W}(t)\right)}\right] = e^{\frac{1}{2}\sigma_I^2(t_i-t)}$$

Then

[1] $\zeta(t) = \zeta I(t)$ and $I(t)$ is F_t-measurable

$$
\begin{aligned}
\Lambda(t) &= \zeta \sum_{i:t_i \geq t} e^{-r(t_i - t)} I(t) e^{\left(\mu_I - \sigma_I \lambda - \frac{\sigma_I^2}{2}\right)(t_i - t)} e^{\frac{1}{2}\sigma_I^2 (t_i - t)} \\
&= \zeta I(t) \sum_{i:t_i \geq t} e^{\left(\mu_I - r - \sigma_I \lambda\right)(t_i - t)}
\end{aligned}
\tag{3.11}
$$

For convenience we place

$$
g(t) = \sum_{i:t_i \geq t} e^{\left(\mu_I - r - \sigma_I \lambda\right)(t_i - t)}
$$

It is concluded that the time-t price of future contributions is

$$
\Lambda(t) = \zeta I(t) g(t) \tag{3.12}
$$

As a result, the precise value of $\Lambda(t)$ is known, and it may be hedged using market assets. The time-t price of the future contributions can then be short-sold at time zero and collect its value $\Lambda(0)$. Rather than waiting for contributions to arrive, the price might be deposited directly into the pension plan. The incoming future payments neutralize the replicating portfolio, and the future payments are treated as if they were already a part of the assets in the portfolio. As a result of using this technique, the wealth process becomes independent of future inflows and introduces a continuous setting by removing the discontinuity caused by payments.

The dynamics of $\Lambda(t)$ should be investigated first in order to derive the explicit form of the replicating portfolio. The risk-neutral (measure **Q**) dynamics of $\Lambda(t)$ between discontinuity points $t_i \; for \; i = 0, 1, 2, \ldots, n$, are provided by formula (3.7).

$$
d\Lambda_t = r\Lambda_t dt + \sigma_I \Lambda_t d\widetilde{W}_t \qquad for \; t \in (t_i, t_{i+1}), \; i = 0, 1, \ldots, n \tag{3.13}
$$

Under **P** (real-word measure) it satisfies

$$
d\Lambda_t = (r + \sigma_I \lambda)\Lambda_t dt + \sigma_I \Lambda_t dW_t \qquad for \; t \in (t_i, t_{i+1}), \; i = 0, 1, \ldots, n \tag{3.14}
$$

The differential $d\Lambda_t$ does not exist at payment periods, and the process development is given by

$$
\Lambda(t_i^+) = \Lambda(t_i) - \zeta(t_i) \qquad for \; i = 0, 1, \ldots, n \tag{3.15}
$$

As a result, at periods t_i, $\Lambda(t_i)$ displays discontinuous dynamics $\forall i$.

In the context of a defined contribution plan, our goal in this part is to provide a pension fund guarantee through continuous-time trading and, in the next section, using the discrete-time trading assumption. Two alternative floor processes with deterministic and stochastic dynamics are developed with this goal in mind.

3.2 Continuous-Time trading

The main aspects of Section 2.5.1, in which the standard CPPI strategy was presented in continuous time, are reported.

A continuous-time securities market is studied with the non-risky component (a money market account) R_t and the risky component (a security or stock exchange index) S_t with price dynamics given by:

$$dR_t = rR_t dt \tag{3.16}$$

and

$$dS_t = \mu_s S_t dt + \sigma_s S_t dW_t \qquad S(0) = S_0 \tag{3.17}$$

W_t is a Brownian motion defined over the entire probability space $(\Omega; F; \mathbf{P})$ with filtration $\{F_t\}_{t\in[0;T]}$; μ_s, σ_s are real constants with $\sigma_s > 0$, and the spot rate r is considered constant. The Black-Scholes market setting is complete in the sense that it can be duplicated by proper self-financing trading methods using measurable random variables F_t sufficiently integrable.

It's important to keep in mind that, thanks to market completeness, future contributions (given by (3.3)) can be fully hedged. As a result, given the stream of future contribution payments $\Lambda(t)$, a replicating portfolio exists. Under the risk-neutral measure, this stream could be seen as a liability, and a unique price can be issued. This replication will be employed in a variety of ways when dealing with portfolio allocation. The precise CPPI techniques under continuous time trading with the new floors are described in this section.

The suitable replicating portfolio should be found to verify that $\Lambda(t)$ is hedgeable. A self-financing trading strategy is explored for this, where $\phi^S; \phi^R; \phi^\Lambda$ represent the number of units held from respective asset, with $\phi^\Lambda = -1$. We indicate with Φ the portfolio approach that takes into account the aforementioned shares. As a result, the strategy should be built in such a way that $\bar{\Phi} = (\phi^S; \phi^R)$ is a $\Lambda(t)$ replicating strategy. The replicating portfolio that follows the Φ strategy is denoted by Y^Φ, and its dynamics are indicated by:

$$\begin{aligned} dY_t^\Phi &= \phi_t^S dS_t + \phi_t^R dR_t - d\Lambda_t \\ &= (\phi_t^S \mu_s S_t + \phi_t^R r R_t - \Lambda_t(r + \sigma_I \lambda))dt + (\phi_t^S \sigma_s S_t - \Lambda_t \sigma_I) dW_t \end{aligned} \tag{3.18}$$

In order to derive the quotas for the risky component S_t and for the riskless component R_t, that identify the hedging portfolio, we have to solve a simple system of two equations and two unknowns:

$$\begin{cases} \phi_t^S \mu_s S_t + \phi_t^R r R_t - \Lambda_t(r + \sigma_I \lambda) = 0 \\ \phi_t^S \sigma_s S_t - \Lambda_t \sigma_I = 0 \end{cases}$$

$$\begin{cases} \phi_t^S \mu_s S_t + \phi_t^R r R_t - \Lambda_t r - \Lambda_t \sigma_I \frac{\mu_s - r}{\sigma_s} = 0 \\ \phi_t^S = \frac{\Lambda_t \sigma_I}{\sigma_s S_t} \end{cases}$$

$$\begin{cases} \frac{\Lambda_t \sigma_I}{\sigma_s}\mu_s + \phi_t^R r R_t - \Lambda_t r - \Lambda_t \sigma_I \frac{\mu_s - r}{\sigma_s} = 0 \\ \phi_t^S = \frac{\Lambda_t \sigma_I}{\sigma_s S_t} \end{cases}$$

$$\begin{cases} \phi_t^R = \frac{\Lambda_t}{R_t}(1 - \frac{\sigma_I}{\sigma_s}) \\ \phi_t^S = \frac{\Lambda_t \sigma_I}{\sigma_s S_t} \end{cases} \tag{3.19}$$

The beneficiary receives the amount

$$\Lambda(0) = \zeta I(0) g(0)$$

at time $t = 0$, which is the original value of the pension portfolio, by short-selling the replicating portfolio $\bar{\Phi}$. Of course, this is a huge increase over the starting premium of

$$\zeta(0) = \zeta I(0)$$

It can be asserted that future payments to the fund may be at risk due to the contributor's prospective unemployment. However, the future payments that are guaranteed by the contributor's employment contract can then be valued and replicated. Because future contributions are now matched by inflows to the replicating portfolio, the (personal) wealth process of the beneficiary invested in the fund continues even during contribution payment periods.

3.2.1 Guarantee types

Two sorts of guarantees/floors, which the CPPI techniques are built on, are introduced in this section. They range from fully stated guarantees based on the beginning net present value of the premium stream to fully specified guarantees based on the actual value of the labor process and updated past guarantees - both of which are stochastic processes.

3.2.2 Net Present Value floor

The following guarantee is fully equivalent to the one previously seen in section 2.5.1, we refer to equations (2.17) and (2.18). We now denote the value of fund with the letter Y (which we had previously, in chapter 1, used for the value of guaranteed benefit), and the value of guarantee to maturity with $\bar{B}$.

The Net Present Value (NPV) floor, a specific instance of deterministic floor, is defined in the current work by making the starting floor value $B(0)$ equal to the net present value of future contributions.

$$B(0) = c\zeta I(0) g(0) \tag{3.20}$$

where c is a constant that represents the guaranteed part of the beginning wealth.

The floor process is assumed to grow at risk-free rate r. As a result, the promised amount paid at maturity T is $B(T) = \bar{B}$:

$$\bar{B} = B(0)e^{rT}$$

After determining the guarantee, the following step is to select the investment, which is defined by the cushion's multiplier m. $C(t)$ is introduced as the difference between the actual value of the defined contribution fund and the floor, that is, $C(t) = Y(t) - B(t)$ as in (2.4). In a CPPI approach, a multiple $mC(t)$ (typically $m >> 1$) is always invested in the stock, with the remainder in the bond. Investor can now change the multiplier m as well as the guarantee level $\bar{B}$. Naturally, a larger guarantee level will reduce the mean of the final value, as a lower value of m. We'll return to the multiplier's ideal choice later.

In this sector of application, we may consider the results obtained in section 2.5.1.Let's consider $\mu = \mu_s$ and $\sigma = \sigma_s$, the value of fund in t F_t becomes the value of performance Y_t and the guarantee to maturity G is now $\bar{B}$

Cushion dynamic

We resume the equations (2.21) and (2.23).

$$\begin{aligned} dC_t &= dY_t - dB_t \\ &= \big(r + m(\mu_s - r)\big)C_t dt + m\sigma_s C_t dW_t \end{aligned} \tag{3.21}$$

The value of cushion at maturity is given by:

$$C_T = C_0 e^{\left(r+m(\mu_s-r)-\frac{m^2\sigma_s^2}{2}\right)T+m\sigma_s W_T} \tag{3.22}$$

Wealth dynamic

Let's recall the equation (2.20).

$$dY_t = \big[mC_t(\mu_s - r) + rY_t\big]dt + \big[m\sigma_s C_t\big]dW_t \tag{3.23}$$

The final wealth $Y(t)$ is defined as the summation of guarantee at maturity and the excess return earned from investing in the initial cushion in stock market.

$$Y_T = \bar{B} + C_0 e^{\left(r+m(\mu_s-r)-\frac{m^2\sigma_s^2}{2}\right)T+m\sigma_s W_T} \tag{3.24}$$

Let's compute the expectation and the variance of Y_T applying the moment generating function definition of Normal distribution.

$$\mathbb{E}_0(Y_T) = \bar{B} + E(C_T) = \bar{B} + C_0 e^{\left(r+m(\mu_s-r)\right)T}$$

$$\mathbb{V}_0(Y_T) = C_0^2 e^{2\left(r+m(\mu_s-r)\right)T}\left(e^{m^2\sigma^2 T} - 1\right)$$

The random floor's dynamics are then investigated, in which the guarantee is defined using future contributions.

3.2.3 Random floor

The random floor is the other floor introduced, which exhibits stochastic dynamics. The goal of current thesis in this CPPI scenario is to guarantee a portion of all previous and future contributions until retirement. Let $\bar{\zeta}(t_i)$ be the guaranteed fraction of the contribution payment at every moment $t_i \in [0; T]$, that is:

$$\bar{\zeta}(t_i) = \bar{\zeta} I(t_i) = c\zeta I(t_i) \qquad \forall i = 0, 1, 2, \ldots, n \tag{3.25}$$

for some real constant $0 < c < 1$. After discounting to time t, the previous payments will be brought forward to the present, and future contributions will be incorporated in the guarantee/floor process. The floor process is defined as the sum of time-t values of contributions $\bar{\zeta}(t_i)$ for $i = 0, 1, 2, \ldots, n$, with $B(t)$ indicating the floor value at time t.

$$B(t) = \sum_{i:0 \leq t_i < t} e^{t(t-t_i)} \bar{\zeta}(t_i) + \mathbb{E}_t^{\mathbf{Q}} \left[\sum_{i:t_i \geq t} e^{-r(t_i - t)} \bar{\zeta}(t_i) \middle| F_t \right] \tag{3.26}$$

We now present the dynamics of the bond floor both in terms of risk neutral and under the real word measure. Applying Itô's formula to $B(t)$:

$$dB(t) = rB(t)dt + c\sigma_I \Lambda(t) d\widetilde{W}(t) \qquad \textit{under } \mathbf{Q} \textit{ measure} \tag{3.27}$$

$$dB(t) = \big(rB(t) + c\sigma_I \lambda \Lambda(t)\big)dt + c\sigma_I \Lambda(t) dW(t) \qquad \textit{under } \mathbf{P} \textit{ measure} \tag{3.28}$$

The goal now is to demonstrate that this newly defined floor method can be hedged (or replicated). Otherwise, the CPPI approach would fail because it would be difficult to replicate floor behavior using assets, making it impossible to constantly evolve above the floor.
Following that, it is demonstrated that $B(t)$ has a replication strategy. A self-financing hedging strategy is considered, with η^S and η^R denoting the number of units held from each asset. Using $\eta^B = -1$ and treating $B(t)$ as a portfolio, it holds:

$$\begin{aligned} dY_t^\eta &= \eta_t^S dS_t + \eta_t^R dR_t - dB_t \\ &= (\eta_t^S \mu_s S_t + \eta_t^R r R_t - rB_t - \sigma_I \lambda \Lambda(t))dt + (\eta_t^S \sigma_s S_t - c\sigma_I \Lambda_t) dW_t \end{aligned} \tag{3.29}$$

Following the same procedure as in chapter 3.2, in order to derive the quotas for the risky component S_t and for the riskless component R_t we solve a simple system of two equations and two unknowns:

$$\begin{cases} \eta_t^S \mu_s S_t + \eta_t^R r R_t - rB_t - \sigma_I \lambda \Lambda(t) = 0 \\ \eta_t^S \sigma_s S_t - c\sigma_I \Lambda_t = 0 \end{cases}$$

$$\begin{cases} \eta_t^S = \frac{\Lambda_t}{S_t} \frac{c\sigma_I}{\sigma_s} \\ \eta_t^R = \frac{1}{R_t} \left(B_t - \Lambda_t \frac{c\sigma_I}{\sigma_s} \right) \end{cases} \tag{3.30}$$

Wealth dynamic The portfolio should be constructed in such a way that it may imitate the floor while constantly being above it. To do so, the asset exposure should contain both the percentage coming from replicating strategy (η^S) and the CPPI exposure, which is mc_t as in (2.9). Therefore, now the exposure is given by:

$$E_t = mC_t + \eta_t^S S_t \tag{3.31}$$

and it provide the total amount invested in the stock index, the remaining funds are invested in the bond.

The wealth dynamic is given by:

$$\begin{aligned}
dY_t &= \frac{E_t}{S_t} dS_t + \frac{Y_t - E_t}{R_t} dR_t \\
&\quad \textit{inserting equations } (3.31) \textit{ and } (3.30) \\
&= \frac{mC_t + \eta_t^S S_t}{S_t}\Big[\mu_s S_t dt + \sigma_s S_t dW_t\Big] + \frac{C_t + B_t - mC_t + \eta_t^S S_t}{R_t}\Big[r R_t dt\Big] \\
&= \frac{mC_t + \Lambda_t \frac{c\sigma_I}{\sigma_s}}{S_t}\Big[\mu_s S_t dt + \sigma_s S_t dW_t\Big] + \frac{C_t + B_t - mC_t + \Lambda_t \frac{c\sigma_I}{\sigma_s}}{R_t}\Big[r R_t dt\Big] \\
&= \Big[mC_t\mu_s + \Lambda_t c\frac{\sigma_I}{\sigma_s}\mu_s + r(C_t + B_t) - mC_t - \Lambda_t c\frac{\sigma_I}{\sigma_s} r\Big]dt + \Big[mC_t\sigma_s + \Lambda_t c\sigma_I\Big]dW_t \\
&= \Big[mC_t(\mu_s - r) + (C_t + B_t)r\Big]dt + \Big[mC_t\sigma_s + \Lambda_t c\sigma_I\Big]dW_t
\end{aligned} \tag{3.32}$$

with initial value $Y_0 = \Lambda_0$.
The portfolio value process must constantly move ahead of the guarantee process, i.e.

$$Y_t \geq B_t \qquad \forall t \in [0; T]$$

Given the condition $0 < c < 1$, one can say that

$$\zeta I(0)g(0) > \bar{\zeta} I(0)g(0)$$

and also

$$\zeta I(t)g(t) > c\zeta I(t)g(t) \qquad \forall t \in [0; T]$$

that is,

$$Y(t) > B(t) \qquad \forall t \in [0; T]$$

. As a result the condition is always satisfied and the cushion value is positive $\forall t \in [0; T]$.
This is because we are in an ideal continuous time trading scenario.

The changes made in this section only impact the bond floor, therefore the cushion dynamics remain unchanged; as can also be seen analytically:

$$\begin{aligned} dC_t &= dY_t - dB_t \\ &\quad \textit{inserting equations (3.28) and (3.32)} \\ &= \Big[mC_t(\mu_s - r) + (C_t + B_t)r\Big]dt + \Big[mC_t\sigma_s + \Lambda_t c\sigma_I\Big]dW_t \\ &\quad - \Big[rB_t + c\sigma_I\lambda\Lambda_t\Big]dt + c\sigma_I\Lambda_t dW_t \\ &= \Big[mC_t(\mu_s - r) + rC_t\Big]dt + m\sigma_s C_t dW_t \end{aligned} \tag{3.33}$$

The value in T is the same as the one calculated above, (3.22).

We now calculate the value at maturity of fund and guarantee with the related expected values. The value in T of the bond floor derives directly from the definition (3.26)

$$\begin{aligned} B_T &= \sum_{i:0\le t_i<T} e^{r(T-t_i)}\bar{\zeta}(t_i) \\ &= \bar{\zeta} \sum_{i:0\le t_i<T} e^{r(T-t_i)} I(t_i) \end{aligned} \tag{3.34}$$

The expectation of B_T

$$\begin{aligned} \mathbb{E}_0(B_T) &= \sum_{i:0\le t_i<T} e^{r(T-t_i)}\mathbb{E}_{t_i}(\bar{\zeta}(t_i)) \\ &= \bar{\zeta} \sum_{i:0\le t_i<T} e^{r(T-t_i)}\mathbb{E}_{t_i}(I(t_i)) \\ &\quad \textit{using the dynamic (3.1) and the result obtained in section 2.5.1} \\ &= \bar{\zeta} I(0) \sum_{i:0\le t_i<T} e^{rT+(\mu_I - r)t_i} \end{aligned} \tag{3.35}$$

Remark 2. The value in 0 of the guarantee is given by:

$$\begin{aligned} B_0 &= \mathbb{E}_t^{\mathbf{Q}}\Big[\sum_{i:t_i\ge 0} e^{-r(t_i)}\bar{\zeta}(t_i)\Big| F_t\Big] \\ &\quad \textit{using the dynamic (3.5)} \\ &= \bar{\zeta} I(0) \sum_{i:t_i\ge 0} e^{\left(\mu_I - r - \lambda\sigma_I\right)t_i} \end{aligned}$$

The terminal wealth Y_T is equal to the maturity guarantee added to the excess return from investing the initial cushion in the stock market.

$$\begin{aligned} Y_T &= B_T + C_T \\ &= \bar{\zeta} \sum_{i:0\le t_i<T} e^{r(T-t_i)} I(t_i) + C_0 e^{\left(r + m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right)T + m\sigma_s W_T} \end{aligned} \tag{3.36}$$

Therefore the expected value is:

$$\begin{aligned}\mathbb{E}_0(Y_T) &= \mathbb{E}_0(B_T) + \mathbb{E}_0(C_T)\\ &= \bar{\zeta} I(0) \sum_{i:0\leq t_i<T} e^{rT+(\mu_I-r)t_i} + C_0 e^{(r+m(\mu_s-r))T}\end{aligned} \tag{3.37}$$

For the sake of completeness, as previously done, we calculate the variance of process Y_T

$$\begin{aligned}\mathbb{V}_0(Y_T) &= \mathbb{V}_0(B_T) + \mathbb{V}_0(C_T)\\ &= \bar{\zeta} \sum_{i:t_i\geq 0} e^{(T-t_i)} I_0^2 e^{2\mu_I t_i} \left(e^{\sigma_I^2 t_i} - 1\right) + C_0^2 e^{2\left(r+m(\mu_s-r)\right)T} \left(e^{m^2\sigma^2 T} - 1\right)\end{aligned} \tag{3.38}$$

3.2.4 Risk measures

Let's look at gap and cash lock risks for the CPPI methods recently discussed in the context of continuous time trading, which were discussed in general at the end of previous chapter. Due to the inability of instantaneous trading, several of these risks emerge under the assumption of discrete-time trading. They can also happen in a market with continuous trading when there is a quick decline in the market, given that the risky component's trajectory is subject to unexpected jumps. The incidence of these risks is now discussed in a continuous-time trading system, revealing the risk measurements involved.

Cash-lock risk

As previously stated, cash lock happens when all of assets are invested in a risk-free asset and remain there until retirement. This means that at some point during the plan, the cushion will be zero, implying that the portfolio's value will be equal to the bond floor. Due to the zero equity exposure in this situation, there is no way for the portfolio's value to recover until retirement. Continuous time trading eliminates the chance of being in this scenario because the portfolio management will react instantly and prevent the cushion from reaching zero. This is also supported by the cushion's continuous dynamics, which are provided using geometric Brownian motion. We remind that the dynamic of cushion is the same in both the Net Present Value and Random Floor cases.

Cash-lock probability The cash lock probability for the period $(t; s)$ is denoted by $\mathbb{P}^{CL}_{t;s}$. It is the probability that exposure compared to the fund's value in s is equal to zero since the proportion of risky activity is equal to γ at a previous instant t, that is

$$\mathbb{P}^{CL}_{t;s} = \mathbb{P}\left(\frac{mC_s}{Y_s} = 0 \middle| \frac{mC_t}{Y_t} = \gamma\right) \tag{3.39}$$

For continuous-time CPPI with NPV and random floors, this $\mathbb{P}^{CL}_{t;s}$ probability is zero, as previously stated. However, this is only true for a continuous-time artificial market; in marketplaces where dynamic trading is not possible, the probability is non-zero.

Gap risk

As previously said, the gap risk is another issue with dynamic portfolio insurance schemes such as CPPI. The risk of not achieving the retirement guarantee with probability one is known as gap risk. Given market assumptions, we have demonstrated that this risk is likewise absent in the case of ordinary continuous-time CPPI. Indeed the market's incompleteness is the source of this risk of shortfall. We can't completely reproduce every contract in discrete trading, thus we're not in a complete market.

Shortfall probability In this scenario we define the probability of shortfall as the probability that the cushion value at maturity becomes negative, which means that the CPPI strategy's value at maturity Y_T is less than the promised amount B_T. Now let's use the $\mathbb{P}_T^{SF}$ notation to define the shortfall probability at maturity:

$$\mathbb{P}_T^{SF} = \mathbb{P}(C_T < 0) \tag{3.40}$$

Again, the cushion cannot be negative at any point throughout the investment period due to continuous trading. The dynamics of the cushion, which are presented in (3.21) for both floors, demonstrate this clearly. It is non-negative since the cushion is Geometric Brownian motion. Because these are significant risks that exist in real-world financial markets, they are explored in the next section in a discrete trading environment.

3.3 Discrete-Time trading

The topic of portfolio insurance for defined contribution pension funds is addressed in this chapter using the discrete-time trading assumption. The market dynamics are considered to be continuous with the same assets as in the preceding section. So, the market is made up of bond and stock, whose dynamics are described in (3.16) and (3.17), respectively. One noteworthy characteristic that emerges for this market is its incompleteness. Because perfect hedging is not attainable in a discrete-time trading environment, the market under consideration becomes incomplete. The participant is supposed to contribute a constant proportion zeta of his work income, analogous to the setup stated in Section 3.1. The income dynamics and contribution rate are shown in (3.1) and (3.3), respectively. Because the market is not complete in this situation, these future contributions cannot be replicated. The overall scenario is identical to that of S. Balder, M. Brandl, A. Mahayni (2009) [3], but the issue is framed in the presence of consecutive random contributions into wealth at specified periods. Due to the inability of instantaneous rebalancing in discrete-time trading, the significant risks of cash-lock and gap risk exist. Furthermore, the CPPI portfolio is not self-financing owing to the input of monthly payments into the portfolio value. It is assumed in this discrete-time framework that trading occurs immediately following the contribution payment $\zeta(t_i)$ at time $t_i \in [0; T]$ for $i = 0, 1, 2, \ldots, n-1$. Let $\vec{t} = \{t_{0=0,t_1,\ldots,t_n}\}$ denote the collection of established payment dates. When trading occurs at time t_i, the number of shares held in the risky asset remains constant until the next trading date, i.e. across the time period $(t_i; t_{i+1}]$. Instead of

considering invested percentages of wealth, which may fluctuate between rebalancing periods, the quantity of assets at time t is taken into consideration and denoted by $N = (N^S, N^R)^2$.
N^S and N^R are the number of stock index and bond units, respectively.
The discrete-time CPPI approach is defined $\forall t \in (t_i, t_{i+1})$ *and* $i = 0, 1, 2, \ldots, n-1$:

$$N_t^S = \max\left\{\frac{mC_{t_i}}{S_{t_i}}; 0\right\}, \quad N_t^R = \frac{Y_{t_i} - N_t^S S_{t_i}}{R_{t_i}} \tag{3.41}$$

It is necessary to keep in mind that the cushion may become temporarily negative owing to the impossibility of instantaneous rebalancing. A limit on the number of risky assets is enforced to prevent the negative asset exposure produced by the negative cushion. We do not permit short positions in the risky asset, and the asset exposure is capped at zero. The discrete-time CPPI, such as the continuous-time CPPI described in Chapter 2, does not incorporate short sale limitations on the riskless asset. Remember that the inclusion of borrowing limits precludes the otherwise stated closed-form answers. However, take note that the risk profile of the simplified version might be utilized as a benchmark for the most realistic instance. The discrete-time valuation process of the defined contribution pension portfolio at time t_i represented by $N_{t_i} = (N_{t_i}^S, N_{t_i}^R)$ is described as follow:

$$Y_t(N_{t_i}) = \begin{cases} N_{t_i}^S S_t + N_{t_i}^R R_t & t \in (t_i, t_{i+1}) \\ N_{t_i}^S S_{t_{i+1}} + N_{t_i}^R R_{t_{i+1}} + \zeta(t_i) & t = t_{i+1} \end{cases} \tag{3.42}$$

The premium payment $\zeta(t_{i+1})$ will be invested in assets an instant later at time t_{i+1}^+, causes a jump in the portfolio value at t_{i+1}. That is

$$Y(t_{i+1})(N_{t_i}) = Y(t_{i+1}^-)(N_{t_i}) + \zeta(t_{i+1}) = Y(t_{i+1}^+)(N_{t_{i+1}}) \tag{3.43}$$

where $N_{t_{i+1}}$ is the portfolio shares after the rebalancing at time t_{i+1}, $\forall i = 0, 1, 2, \ldots, n-1$.

3.3.1 Guarantee types

This section discusses floor variations in discrete-time trading and the CPPI methods that go with them. The net present value and random floor methods are introduced first.

3.3.2 Net Present Value floor

The first mentioned floor is the net present value (NPV), which is initially explained in previous section for a continuous-time trading context. Instead of defining a constant guarantee amount for time T, the initial floor in the CPPI approach with the NPV floor is set to reflect the discounted value of future contributions. This starting floor, represented by $B(0)$, is supposed to increase at the risk-free rate r until maturity. The expectation of future payments is estimated next to understand the dynamics of this deterministic floor. Let $\Lambda(t)$ represent the market price at time

[2] Let's go back to the notation used in Chapter 2

t of the stream of future contributions payable between time t and time T. Then you have,

$$\Lambda(t) = \mathbb{E}_t^{\mathbf{Q}^*}\left[\sum_{i:t_i\geq t} e^{-r(t_i-t)}\zeta(t_i)\middle|F_t\right] = \zeta I(t)g(t) \tag{3.44}$$

where $g(t)$ is defines a sin section 3.1.
The difference between this equation and (3.7) is in the probability measure. We highlighted the market's incompleteness at the beginning of section owing to the impossibility of replicating all securities in the setting of discrete-time trading. In such models, multiple risk-neutral measures are often included; in practical implementations, one of these risk-neutral measures is selected and utilized for pricing. The risk-neutral measures selected for this purpose are determined by how the model is designed and calibrated. Here $\mathbf{Q}^*$ denotes the corresponding martingale measure for arbitrage free pricing.[3]
As a result the dynamics of labor income $I(t)$ and of the stream of future contributions $\Lambda(t)$ under this martingale measure $\mathbf{Q}^*$ is given respectively by:

$$dI_t = (\mu_I - \sigma_I\lambda)I_t dt + \sigma_I I_t d\bar{W}_t \tag{3.45}$$

$$d\Lambda_t = r\Lambda_t dt + \sigma_I \Lambda_t d\bar{W}_t \tag{3.46}$$

where $\bar{W}_t$ is the Brownian motion under $\mathbf{Q}^*$and λ is the market price of risk defined in (3.6). Under risk natural measure $\mathbf{P}$ these dynamics remains obviously the same.

As for the context of continuous time trading, the NPV at evaluation time started $t = 0$ is equal to a guaranteed constant fraction c of the total amount of the contributions:

$$B(0) = c\Lambda(0) = c\zeta I(0)g(0)$$

and at maturity T:

$$\bar{B} = B(0)e^{rT}$$

Hence the dynamics of bond floor is:

$$dB(t) = rB(t)dt, \qquad B(0) = c\Lambda(0) \tag{3.47}$$

Wealth dynamics

We said that in this discrete-time framework, trading occurs immediately after the contribution payment $\zeta(t_i)$ at time $t_i \in [0;T]$ for $i = 0, 1, \ldots, n-1$. Indeed in this context we must consider the possibility that the cushion becomes negative between two rebalancing instants. Furthermore, to express the portfolio value, we will have

[3] The fundamental idea behind no-arbitrage pricing is to reproduce the payoff of a derivative security by trading in the underlying asset (which we call a stock) and the money market account. Steven E. Shreve (2004) [35]

to consider two cases; first when the evaluation time t is between two instants in which the contributions are paid, therefore for $t \in (t_i; t_{i+1})$:

$$Y_t = \begin{cases} mC(t_i)\frac{S_t}{S_{t_i}} + \big(Y(t_i) - mC(t_i)\big)\frac{R_t}{R_{t_i}} & C(t_i) > 0 \\ Y(t_i)\frac{R_t}{R_{t_i}} & C(t_i) \leq 0 \end{cases} \tag{3.48}$$

and when the evaluation date t is an instant of contribution payment, that is $t = t_{i+1}$:

$$Y_t = \begin{cases} mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + \big(Y(t_i) - mC(t_i)\big)\frac{R_{t_{i+1}}}{R_{t_i}} + \zeta(t_{i+1}) & C(t_i) > 0 \\ Y(t_i)\frac{R_{t_{i+1}}}{R_{t_i}} + \zeta(t_{i+1}) & C(t_i) \leq 0 \end{cases} \tag{3.49}$$

One can easy obtains the wealth dynamics:

for $t \in (t_i, t_{i+1})$

$$dY_t = \begin{cases} [mC_{t_i}(\mu_s - r) + rY_{t_i}]dt + [m\sigma_s C_{t_i}]dW_{t_i} & C(t_i) > 0 \\ rY(t_i)dt & C(t_i) \leq 0 \end{cases} \tag{3.50}$$

and when $t = t_{i+1}$:

$$Y(t_{i+1}) = Y(t_i^-) + \zeta(t_i) \tag{3.51}$$

Cushion dynamics

Given that $C(t) = Y(t) - B(t)$, it follows:
for $t \in (t_i, t_{i+1})$

$$C_t = \begin{cases} C(t_i)\Big(m\frac{S_t}{S_{t_i}} + (1-m)e^{r(t-t_i)}\Big) & C(t_i) > 0 \\ C(t_i)e^{r(t-t_i)} & C(t_i) \leq 0 \end{cases} \tag{3.52}$$

and the dynamic is given by:

$$dC_t = \begin{cases} [mC_{t_i}(\mu_s - r) + rC_{t_i}]dt + [m\sigma_s C_{t_i}]dW_{t_i} & C(t_i) > 0 \\ rC(t_i)dt & C(t_i) \leq 0 \end{cases} \tag{3.53}$$

At payment times $t = t_{i+1}$, $\forall i = 0, 1, 2, \dots, n-1$, the differential does not exist and the process evolution is given by

$$C(t_{i+1}) = C(t_{i+1}^-) + \zeta(t_{i+1}) \tag{3.54}$$

It should be noted from the dynamics that in order to always be above the floor, $m \leq 1$ must be set. However, keep in mind that CPPI techniques are typically formulated with a multiplier greater than one. Otherwise, in a declining financial market, there is a non-zero possibility of negative cushion. One of the key distinctions between our framework and the traditional CPPI is that the cushion process only turns negative momentarily. With the assistance of contribution inflows, the portfolio has the potential to recover and regain a positive cushion following a negative surplus.

3.3.3 Random floor

The randomness of the labor income process is significant in the evolution of the portfolio in this DC pension context. Because contribution payments are put into the portfolio wealth at regular intervals, it is only natural to increase the floor process by the same increments for insurance. As a result, the random floor is introduced, which not only grows at the risk-free rate between inter-payment periods, but also makes jumps at consecutive payment times, as the value process does. A portion of each contribution is guaranteed, as was done in the NPV floor example. To be more specific, $c\zeta(t_i)$ is included in the floor for some real constant $0 < c < 1$ for each payment $\zeta(t_i)$ made at time $t_i[0;T]$.Therefore, jumps in the floor process are fractions of the inflows into wealth. The random floor is defined in as the sum of time-value of paid contributions. Payments are paid at predetermined trading dates $t_i \in [0;T]$, and the collected amount rises at the risk-free rate r between payment periods. At time t, the guarantee is defined as

$$B_t = \begin{cases} \sum_{k=0}^{i} e^{r(t-t_k)} c\zeta(t_k) & t \in (t_i, t_{i+1}) \\ B(t_{i+1}^-) + c\zeta(t_{i+1}) & t = t_{i+1} \end{cases} \tag{3.55}$$

Hence, between payment dates $B(t)$ has the dynamics

$$dB(t) = rB(t)dt, \qquad B(0) = c\zeta(0) \tag{3.56}$$

and for $t = t_i, \ \forall i = 0, 1, 2, \ldots, n$

$$B(t_i) = B(t_i^-) + c\zeta(t_i) \tag{3.57}$$

As a consequence we can rewrite (3.55)

$$B(t) = \begin{cases} e^{r(t-t_i)} B(t_i) & t \in (t_i, t_{i+1}) \\ B(t_{i+1}^-) + c\zeta(t_{i+1}) & t = t_{i+1} \end{cases} \tag{3.58}$$

Wealth dynamics

We can rewrite the dynamics of the performance value in the same way as done in the previous section for the NPV floor.
For $t \in (t_i; t_{i+1})$:

$$Y_t = \begin{cases} mC(t_i)\frac{S_t}{S_{t_i}} + \left(Y(t_i) - mC(t_i)\right)\frac{R_t}{R_{t_i}} & C(t_i) > 0 \\ Y(t_i)\frac{R_t}{R_{t_i}} & C(t_i) \leq 0 \end{cases} \tag{3.59}$$

and when $t = t_{i+1}$:

$$Y_t = \begin{cases} mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + \left(Y(t_i) - mC(t_i)\right)\frac{R_{t_{i+1}}}{R_{t_i}} + \zeta(t_{i+1}) & C(t_i) > 0 \\ Y(t_i)\frac{R_{t_{i+1}}}{R_{t_i}} + \zeta(t_{i+1}) & C(t_i) \leq 0 \end{cases} \tag{3.60}$$

Cushion dynamics

We derive easily the cushion dynamics for $t \in (t_i; t_{i+1})$ from (3.59) and (3.58):

$$C_t = \begin{cases} C(t_i)\left(m\frac{S_t}{S_{t_i}} + (1-m)e^{r(t-t_i)}\right) & C(t_i) > 0 \\ C(t_i)e^{r(t-t_i)} & C(t_i) \le 0 \end{cases} \tag{3.61}$$

with

$$\begin{aligned} C(t_i) &= Y(t_i^-)(N_{t_i}) + \zeta(t_i) - B(t_i^-) - c\zeta(t_i) \\ &= C(t_i^-) + (1-c)\zeta(t_i) \qquad\qquad \forall i = 0, 1, 2, \ldots, n \end{aligned} \tag{3.62}$$

Remark 3. If we set $c = 1$ the floor is increased by the full contribution amount at each payment time As a consequence the cushion process has no discontinuity at payment times. Indeed relation (3.62) implies that for $c = 1$,

$$C(t) = C(t^-) \qquad \forall t \in [0; T] \tag{3.63}$$

Again, given the dynamics (3.59), (3.60), and (3.61), one should think that for $m \le 1$, the value process always evolves above the floor.[4] Otherwise, if the market drops suddenly, the cushion may become negative. In actual markets, the CPPI methods that have been offered thus far have several significant flaws. When the market rises, one possible difficulty develops in terms of performance. When the price of risky assets rises, resulting in an increase in portfolio value, the floor may become negligible, compromising the profits. When the cushion value becomes so low, another risk develops. In this instance, the entire wealth risks being fully invested in the risk-free asset and remaining below the floor until maturity. Furthermore, because rebalancing is done at discrete intervals rather than continually in actuality, there is a possibility of cushion going negative between two rebalancing dates. Boulier and Kanniganti (2005) [11] propose various CPPI improvements to address these issues. These updated techniques include features that allow the floor to fluctuate in response to strong market conditions, resulting in a path-dependent structure.

3.3.4 Risk measures

In this part, certain risk measures for the CPPI techniques under consideration are computed and examined. We take into account the existing literature in this regard, mainly S. Balder, M.Brandl, A. Mahayni (2009) [3] and R. Korn; A. S. Selcuk-Kestel; B. Z. Temocin (2017) [26].
Aside from comparing different floor cases, the main result is a comparison of financial positions taken at the start of the pension plan. In order to achieve this goal, the portfolio performance is evaluated when a replicating portfolio for future contributions is short sold.

[4] However, in this case, we would definitely acquire very low returns; hence, the CPPIs are defined with $m > 1$.

The probability of the sum of log-normal random variables is encountered in the computation of the risk measures to be given. While there is no closed-form equation for a log-normal cumulative probability density function, there are well-known analytical approximation approaches in the literature. The current study adopts the Fenton-Wilkinson technique (see Appendix A A.3), which involves approximating the log-normal sum with a single log-normal random variable and using moments to estimate the parameters of the new log-normal distribution, as is commonly done in pricing Asian and basket options. As discussed in the subsections, the precise values for the pre-event probability are obtained, and the more sophisticated post-event probabilities are estimated. Following the collection of risk measurements, a numerical example is shown, and the effectiveness of CPPI techniques with NPV and random floor is addressed.

Cash-lock risk

The proposed pension architecture involves regular contributions into the fund at predetermined intervals. These inflows, which substantially alter portfolio allocation, act as a recover from a potential cash-locked position. Keep in mind that cash-locked position is the situation in which the entire wealth has to be invested in the riskless asset until the investment horizon as the only way to guarantee a final wealth above the floor. Because of the strategy's path-dependent nature, the cash-lock probability analysis must be performed for each period, that is for each inter-payment intervals. As a result, local dynamics are investigated first.

The definition of local cash-lock probability for the interval $(t_i, t_{i+1}]$ follows.

Local cash-lock probability The probability for the period $(t_i, t_{i+1}]$ that proportion of the risky asset at t_{i+1} is less than ϵ given that it is equal to γ at t_i, is called γ-ϵ *cash-lock probability* and it is given by:

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} = \mathbb{P}\left(\frac{mC(t_{i+1})}{Y(t_{i+1})} \leq \epsilon \,\middle|\, \frac{mC(t_i)}{Y(t_i)} = \gamma\right) \tag{3.64}$$

for any $i = 0, 1, 2, \ldots, n-1$ and where $0 \leq \gamma \leq 1$. In particular for $\epsilon = 0$ we call (3.64) the *local cash-lock probability.*

It is important to note that when the cushion reaches zero or is negative, the proportion of the portfolio value invested in the risky asset, that is the exposure, is reset to zero at the next trading date. While a cash-lock is impossible in a continuous-time environment, it can occur in our scenario because we assume discrete-time trading and hence cannot promptly react to a negative development of the risky asset. If $\gamma > 0$, the portfolio contains a non-zero position in the risky asset. As a result, calculating the local cash-lock probability becomes more difficult since we now need the distribution of a sum of two correlated lognormal random variables, the incoming premium and the CPPI-performance. The cash-lock probability is studied for two different situations due to the inflow payments made into the wealth. The first is when the payment at time t_{i+1} has not yet arrived, and the second is when it has just arrived. Because these incoming payments contain exogenous randomness, calculating the cash-lock probability becomes more complicated in the latter situation. In the first situation, the cash-lock probability may be thought of as an upper bound for the local cash-lock probability. This probability's upper bound is defined as

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t^-_{i+1}} = \mathbb{P}\Bigg(\frac{mC(t^-_{i+1})}{Y(t^-_{i+1})} \leq \epsilon \Bigg| \frac{mC(t_i)}{Y(t_i)} = \gamma\Bigg) \tag{3.65}$$

for any $i = 0, 1, 2, \ldots, n-1$. This is the *pre-premium cash-lock probability* for$\epsilon = 0$. Specifically, the cash-lock probability for inter-payment periods is determined by assuming that the payment at the beginning has already been made and that the payment at the end of the period has not yet arrived. It is obvious that for $\gamma \leq 0$, the upper bound is 1 since the CPPI prohibits any investment in risky assets during the specified period. As a result, the probability (3.65) gives some information only when $\gamma > 0$. The strategy's dynamics, as shown in (3.41), do not permit investing in risky assets at time t_{i+1} if $C(t^-_{i+1}) = 0$. As a result, cash-lock occurs in this instance if the incoming payment fails to make the cushion positive.

Cash-lock risk in NPV floor The cash-lock risk of the CPPI approach with the NPV floor is examined in this subsection.

Upper bound of local cash-lock probability Considering $\epsilon \geq 0$, $\gamma > 0$ and $m > 1$, an upper bound for the γ-ϵ *cash-lock probability* in NPV floor CPPI is given by

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t^-_{i+1}} = \Phi\left(\frac{\ln\left(\frac{\epsilon(m-\gamma)}{m\gamma(m-\epsilon)} - \frac{1-m}{m}\right) - \left(\mu_S - r - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}\right) \tag{3.66}$$

for the period $(t_i, t_{i+1}]$ with $i = 0, 1, 2, \ldots, n-1$ and where Φ is the cumulative distribution function of standard normal distribution. The upper bound of local cash-lock probability is given by setting ϵ equal to 0:

$$\mathbb{P}^{CL_{\gamma,0}}_{t_i,t^-_{i+1}} = \Phi\left(\frac{\ln\left(\frac{m-1}{m}\right) - \left(\mu_S - r - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}\right) \tag{3.67}$$

Remark 4. Extreme situations occur in the following cases.
In the case of $m \leq 1$ and $\gamma > 0$:

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t^-_{i+1}} = 0$$

In the case of $\gamma < 0$:

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t^-_{i+1}} = 1$$

Proof. First, we calculate the upper bound of cash-lock probability for $\gamma > 0$.
We have the following pre-payment dynamics as a result of (3.49), (3.51) and (3.52):

$$\begin{aligned} Y(t^-_{i+1}) &= mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + \big(Y(t_i) - mC(t_i)\big)\frac{R_{t_{i+1}}}{R_{t_i}} \\ &= mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + \big(Y(t_i) - mC(t_i)\big)e^{r(t_{i+1}-t_i)} \end{aligned}$$

$$C(t_{i+1}^-) = C(t_i)\Big(m\frac{S_{t_{i+1}}}{S_{t_i}} + (1-m)e^{r(t_{i+1}-t_i)}\Big)$$

When these relationships are inserted into the definition of upper bound cash-lock probability provided in (3.65), the following equations result:

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}^-} = \mathbb{P}\Bigg(\frac{mC(t_i)\Big(m\frac{S_{t_{i+1}}}{S_{t_i}} + (1-m)e^{r(t_{i+1}-t_i)}\Big)}{mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + \big(Y(t_i) - mC(t_i)\big)e^{r(t_{i+1}-t_i)}} \leq \epsilon \Bigg| \frac{mC(t_i)}{Y(t_i)} = \gamma\Bigg)$$

$$\textit{we consider the time step constant}: \ t_{i+1} - t_i = \frac{T}{n}$$

$$\textit{and knowing that } Y(t) = C(t) + B(t) \textit{ we have}:$$

$$= \mathbb{P}\Bigg(C(t_i)(m-\epsilon)(m\frac{S(t_{i+1})}{S(t_i)} + (1-m)e^{r\frac{T}{n}}) \leq B(t_i)e^{r\frac{T}{n}}\epsilon \Bigg| C(t_i) = \frac{\gamma B(t_i)}{m-\gamma}\Bigg)$$

$$= \mathbb{P}\Bigg(\frac{\gamma B(t_i)}{m-\gamma}(m-\epsilon)(m\frac{S(t_{i+1})}{S(t_i)} + (1-m)e^{r\frac{T}{n}}) \leq B(t_i)e^{r\frac{T}{n}}\epsilon\Bigg)$$

$$= \mathbb{P}\Bigg(\frac{S(t_{i+1})}{S(t_i)} \leq \frac{\epsilon e^{r\frac{T}{n}}(m-\gamma)}{m\gamma(m-\epsilon)} - \frac{(1-m)e^{r\frac{T}{n}}}{m}\Bigg)$$

Inside the last probability we take logarithm of both sides and remembering (2.15), we have:

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}^-} = \mathbb{P}\Bigg((\mu_S - \frac{\sigma_S^2}{2})\frac{T}{n} + \sigma_S(W(t_{i+1}) - W(t_i)) \leq r\frac{T}{n} + \ln\Big(\frac{\epsilon(m-\gamma)}{\gamma(m-\epsilon)m} - \frac{1-m}{m}\Big)\Bigg)$$

$$= \mathbb{P}\Bigg(W(t_{i+1}) - W(t_i) \leq \frac{1}{\sigma_S}\Bigg(\ln\Big(\frac{\epsilon(m-\gamma)}{\gamma(m-\epsilon)m} - \frac{1-m}{m}\Big) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}\Bigg)\Bigg)$$

Since Brownian increments have normal distribution, i.e.$W(t_{i+1}) - W(t_i) \sim N(0, \frac{T}{n})$, it follows that

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}^-} = \Phi\Bigg(\frac{\ln\big(\frac{\epsilon(m-\gamma)}{m\gamma(m-\epsilon)} - \frac{1-m}{m}\big) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}\Bigg)$$

which completes the proof for $\gamma > 0$.

It is clear that the upper bound for this probability is equal to 1 for $\gamma \leq 0$ and the lower bond is 0 for $m \leq 1$ and $\gamma > 0$. □

Local cash-lock probability Considering $\epsilon \geq 0$, $\gamma \leq 0$ and $m > 1$, the γ-ϵ *cash-lock probability* in NPV floor CPPI for the period $(t_i, t_{i+1}]$ with $i = 0, 1, 2, \ldots, n-$

1 is given by

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} = \Phi\left(\frac{\ln\left(\frac{B(t_i)m(\epsilon-\gamma)}{\zeta(t_i)(m-\epsilon)(m-\gamma)}\right) - (\mu_I - r - \frac{\sigma_I^2}{2})\frac{T}{n}}{\sigma_I\sqrt{\frac{T}{n}}}\right) \tag{3.68}$$

indeed for $\gamma > 0$ we have to use the approximation method of Fenton and Wilkinson and we arrive to the following result:

$$\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} \cong \Phi\left(\frac{\ln\left(\frac{d(t_i)\sqrt{h(a(t_i),b(t_i))}}{f^2(a(t_i),b(t_i))}\right)}{\sqrt{\ln\left(\frac{h(a(t_i),b(t_i))}{f^2(a(t_i),b(t_i))}\right)}}\right) \tag{3.69}$$

where f and h are functions on $\mathbb{R}^2$ defined as

$$f(x,y) = xe^{\frac{1}{2}\sigma_S^2\frac{T}{n}} + ye^{\frac{1}{2}\sigma_I^2\frac{T}{n}}$$

$$h(x,y) = x^2e^{2\sigma_S^2\frac{T}{n}} + y^2e^{2\sigma_I^2\frac{T}{n}} + 2xye^{\frac{1}{2}(\sigma_S^2+\sigma_I^2)\frac{T}{n}}$$

with

$$a(t_i) = B(t_i)\gamma m\frac{(m-\epsilon)}{m-\gamma}e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}$$

$$b(t_i) = \zeta(t_i)(m-\epsilon)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n}}$$

$$d(t_i) = B(t_i)e^{r\frac{T}{n}}\left(\epsilon - \frac{\gamma(m-\epsilon)(1-m)}{m-\gamma}\right)$$

$\forall\ i = 0,1,2,\ldots,n-1$.
Setting $\epsilon = 0$ in both cases we obtain the *cash-lock probability.*
The cash-lock probability is 0 in the case of $\gamma > 0$ if we have $0 \leq m \leq 1$.

Proof. First, we calculate the the γ-ϵ *cash-lock probability* for $\gamma \leq 0$, in this case we have that the cushion in pre-payment instant t_i is negative or zero $C(t_i \leq 0)$. We have the following post-payment dynamics as a result of (3.49) and (3.51),(3.52) and (3.54) (looking at $C(t_i \leq 0)$):

$$Y(t_{i+1}) = Y(t_i)e^{r(t_{i+1}-t_i)} + \zeta(t_{i+1})$$

$$C(t_{i+1}) = C(t_i)e^{r(t_{i+1}-t_i)} + \zeta(t_{i+1})$$

Considering the time step constant $(t_{i+1} - t_i = \frac{T}{n})$ the probability is given as:

$$
\begin{aligned}
\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} &= \mathbb{P}\bigg(\frac{mC(t_i)e^{r\frac{T}{n}} + m\zeta(t_{i+1})}{Y(t_i)e^{r\frac{T}{n}} + \zeta(t_{i+1})} \le \epsilon \bigg| C(t_i) = \frac{\gamma B(t_i)}{m-\epsilon}\bigg) \\
&= \mathbb{P}\bigg((C(t_i)(m-\epsilon) - \epsilon B(t_i))e^{r\frac{T}{n}} \le (\epsilon - m)\zeta(t_{i+1}) \bigg| C(t_i = \frac{\gamma B(t_i)}{m-\gamma})\bigg) \\
& \textit{using } (3.2) \\
&= \mathbb{P}\bigg((m-\epsilon)\zeta I(t_i) e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I(W(t_{i+1}) - W(t_i))} \le \Big(\epsilon - \frac{\gamma(m-\epsilon)}{m-\gamma}\Big) B(t_i) e^{r\frac{T}{n}}\bigg) \\
& \textit{taking logarithm of both side} \\
&= \mathbb{P}\bigg((\mu_I - \frac{\sigma_I^2}{2})\frac{T}{n} + \sigma_I(W(t_{i+1}) - W(t_i)) \le \ln\bigg(\frac{\Big(\epsilon - \frac{\gamma(m-\epsilon)}{m-\gamma}\Big)B(t_i)e^{r\frac{T}{n}}}{(m-\epsilon)\zeta(t_i)}\bigg)\bigg) \\
&= \mathbb{P}\bigg(W(t_{i+1}) - W(t_i) \le \frac{\ln\Big(\frac{m(\epsilon-\gamma)B(t_i)}{(m-\gamma)(m-\epsilon)\zeta(t_i)}\Big) - (\mu_I - r - \frac{\sigma_I^2}{2})\frac{T}{n}}{\sigma_I}\bigg) \\
&= \Phi\Bigg(\frac{\ln\Big(\frac{B(t_i)m(\epsilon-\gamma)}{\zeta(t_i)(m-\epsilon)(m-\gamma)}\Big) - (\mu_I - r - \frac{\sigma_I^2}{2})\frac{T}{n}}{\sigma_I\sqrt{\frac{T}{n}}}\Bigg)
\end{aligned} \tag{3.70}
$$

which completes the first part of the proof.

Now, we find the γ-ϵ *cash-lock probability* for $\gamma > 0$.
The post-payment dynamics follow from (3.49) and (3.51),(3.52) and (3.54) (looking at $C(t_i > 0)$):

$$
\begin{aligned}
Y(t_{i+1}) &= mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + \big(Y(t_i) - mC(t_i)\big)e^{r(t_{i+1}-t_i)} + \zeta(t_{i+1}) \\
C(t_{i+1}) &= C(t_i)\Big(m\frac{S_{t_{i+1}}}{S_{t_i}} + (1-m)e^{r(t_{i+1}-t_i)}\Big) + \zeta(t_{i+1})
\end{aligned}
$$

Substituting these relations into the definition (3.64), we obtain:

$$
\begin{aligned}
\mathbb{P}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} &= \mathbb{P}\Bigg(\frac{mC(t_i)\Big(m\frac{S(t_{i+1})}{S(t_i)}+(1-m)e^{r\frac{T}{n}}\Big)+\zeta(t_{i+1})}{mC(t_i)\frac{S(t_{i+1})}{S(t_i)}+(Y(t_i)-mC(t_i))e^{r\frac{T}{n}}+\zeta(t_{i+1})}\le\epsilon\Bigg|C(t_i)=\frac{\gamma B(t_i)}{m-\gamma}\Bigg)\\
&= \mathbb{P}\Bigg(C(t_i)\Big(m(m-\epsilon)\frac{S(t_i)}{S(t_i)}+(m-\epsilon)(1-m)e^{r\frac{T}{n}}\Big)\\
&\qquad \le(\epsilon-m)\zeta(t_{i+1})+\epsilon B(t_i)e^{r\frac{T}{n}}\Bigg|C(t_i)=\frac{\gamma B(t_i)}{m-\gamma}\Bigg)\\
&= \mathbb{P}\Bigg(\frac{\gamma B(t_i)}{m-\gamma}\Big(m(m-\epsilon)\frac{S(t_i)}{S(t_i)}+(m-\epsilon)(1-m)e^{r\frac{T}{n}}\Big)-\epsilon B(t_i)e^{r\frac{T}{n}}\\
&\qquad \le(\epsilon-m)\zeta(t_{i+1})\Bigg)\\
&= \mathbb{P}\Bigg(\frac{\gamma B(t_i)}{m-\gamma}\Big(m(m-\epsilon)\frac{S(t_i)}{S(t_i)}+(m-\epsilon)(1-m)e^{r\frac{T}{n}}\Big)-\epsilon B(t_i)e^{r\frac{T}{n}}\\
&\qquad \le(\epsilon-m)\zeta I(t_i)e^{\left(\mu_I-\frac{\sigma_I^2}{2}\right)\frac{T}{n}+\sigma_I(W(t_{i+1})-W(t_i))}\Bigg)\\
&= \mathbb{P}\Bigg(\frac{\gamma m(m-\epsilon)B(t_i)}{m-\gamma}e^{\left(\mu_S-\frac{\sigma_S^2}{2}\right)\frac{T}{n}+\sigma_S(W(t_{i+1})-W(t_i))}+\\
&\qquad +(m-\epsilon)\zeta I(t_i)e^{\left(\mu_I-\frac{\sigma_I^2}{2}\right)\frac{T}{n}+\sigma_I(W(t_{i+1})-W(t_i))}\\
&\qquad \le B(t_i)e^{r\frac{T}{n}}\Big(\epsilon-\frac{(1-m)(m-\epsilon)\gamma}{m-\gamma}\Big)\Bigg)
\end{aligned}
\tag{3.71}
$$

We denote the random variable in the probability as:

$$x_1 = e^{\sigma_S(W(t_{i+1})-W(t_i))}$$

$$x_2 = e^{\sigma_I(W(t_{i+1})-W(t_i))}$$

At this point, knowing $W(t_{i+1})-W(t_i)\sim N(0,\frac{T}{n})$, it is clear that x_1 and x_2 have log-normal distribution:

$$x_1 \sim LogN(0,\sigma_S^2\frac{T}{n})$$

$$x_2 \sim LogN(0,\sigma_I^2\frac{T}{n})$$

Using the following notation

$$a(t_i) = B(t_i)\gamma m\frac{(m-\epsilon)}{m-\gamma}e^{\left(\mu_S-\frac{\sigma_S^2}{2}\right)\frac{T}{n}}$$

$$b(t_i) = \zeta(t_i)(m-\epsilon)e^{\left(\mu_I-\frac{\sigma_I^2}{2}\right)\frac{T}{n}}$$

$$d(t_i) = B(t_i)e^{r\frac{T}{n}}\left(\epsilon - \frac{(1-m)(m-\epsilon)\gamma}{m-\gamma}\right)$$

the searched probability (3.71) takes the form

$$\mathbb{P}(a(t_i)x_1 + b(t_i)x_2 \le d(t_i))$$

We have the probability density function of a sum of two lognormal random variables, but there is no explicit distribution for this summation. The Fenton-Wilkinson approximation assumes that the sum of independent log-normal random variables has a log-normal distribution. In this situation, assuming $X = a(t_i)x_1 + b(t_i)x_2$, the FW technique asserts that:

$$X = e^Y, \quad where\ Y \sim N(\mu_X, \sigma_X^2)$$

As a consequence $X \sim LogN(\mu_X, \sigma_X^2)$, where the mean and the variance of this approximated distribution are:

$$\mu_X = \ln\left(\frac{(a(t_i)e^{\frac{1}{2}\sigma_S^2\frac{T}{n}} + b(t_i)e^{\frac{1}{2}\sigma_I^2\frac{T}{n}})^2}{(a(t_i)^2e^{2\sigma_S^2\frac{T}{n}} + b(t_i)^2e^{2\sigma_I^2\frac{T}{n}} + 2a(t_i)b(t_i)e^{\frac{1}{2}(\sigma_S^2+\sigma_I^2)\frac{T}{n}})^{\frac{1}{2}}}\right)$$

$$\sigma_X^2 = \ln\left(\frac{a(t_i)^2e^{2\sigma_S^2\frac{T}{n}} + b(t_i)^2e^{2\sigma_I^2\frac{T}{n}} + 2a(t_i)b(t_i)e^{\frac{1}{2}(\sigma_S^2+\sigma_I^2)\frac{T}{n}}}{(a(t_i)e^{\frac{1}{2}\sigma_S^2\frac{T}{n}} + b(t_i)e^{\frac{1}{2}\sigma_I^2\frac{T}{n}})^2}\right)$$

We can now approximate the γ-ϵ *cash-lock probability* for $\gamma > 0$ as follow:

$$\begin{aligned}\mathbb{P}((a(t_i)x_1 + b(t_i)x_2 \le d(t_i)) &\cong \mathbb{P}(X \le d(t_i))\\ &= \mathbb{P}(e^Y \le d(t_i))\\ &= \mathbb{P}(Y \le \ln(d(t_i)))\\ &= \mathbb{P}\left(\frac{Y-\mu_X}{\sigma_X} \le \frac{\ln(d(t_i)) - \mu_X}{\sigma_X}\right)\\ &= \Phi\left(\frac{\ln(d(t_i)) - \mu_X}{\sigma_X}\right)\end{aligned}$$

By introducing the notation

$$f(a(t_i), b(t_i)) = a(t_i)e^{\frac{1}{2}\sigma_S^2\frac{T}{n}} + b(t_i)e^{\frac{1}{2}\sigma_I^2\frac{T}{n}}$$

$$h(a(t_i), b(t_i)) = a(t_i)^2e^{2\sigma_S^2\frac{T}{n}} + b(t_i)^2e^{2\sigma_I^2\frac{T}{n}} + 2a(t_i)b(t_i)e^{\frac{1}{2}(\sigma_S^2+\sigma_I^2)\frac{T}{n}}$$

we found

$$\mathbb{P}_{t_i,t_{i+1}}^{CL_{\gamma,\epsilon}} \cong \Phi\left(\frac{\ln\left(\frac{d(t_i)\sqrt{h(a(t_i),b(t_i))}}{f^2(a(t_i),b(t_i))}\right)}{\sqrt{\ln\left(\frac{h(a(t_i),b(t_i))}{f^2(a(t_i),b(t_i))}\right)}}\right)$$

which concludes the proof.

□

Cash-lock risk in random floor The following paragraphs provide an upper bound for cash-lock probability and cash lock probability for different cases of γ in random floor CPPI.

Upper bound of local cash-lock probability Considering $\epsilon \geq 0$, $\gamma > 0$ and $m > 1$, an upper bound for the γ-ϵ *cash-lock probability* in random floor CPPI is given by

$$\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}^-} = \Phi\left(\frac{\ln\left(\frac{\epsilon(m-\gamma)}{m\gamma(m-\epsilon)} - \frac{1-m}{m}\right) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}\right) \tag{3.72}$$

for the period $(t_i, t_{i+1}]$ with $i = 0, 1, 2, \ldots, n-1$ and where Φ is the cumulative distribution function of standard normal distribution. The upper bound of local cash-lock probability is given by setting ϵ equal to 0.

Proof. In section 3.3.3 we have seen that the pre-payment dynamics are identical to those analyzed in the case of CPPI with NPV floor; indeed from (3.59) and (3.61) we have for $\gamma > 0$ (that is for $C(t_i) > 0$):

$$\begin{aligned} Y(t_{i+1}^-) &= mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + \big(Y(t_i) - mC(t_i)\big)\frac{R_{t_{i+1}}}{R_{t_i}} \\ &= mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + \big(Y(t_i) - mC(t_i)\big)e^{r(t_{i+1}-t_i)} \end{aligned}$$

$$C(t_{i+1}^-) = C(t_i)\left(m\frac{S_{t_{i+1}}}{S_{t_i}} + (1-m)e^{r(t_{i+1}-t_i)}\right)$$

Obviously, when these relationships are inserted into the definition of upper bound γ-ϵ cash-lock probability provided in (3.65), we have the same result seen for NPV floor:

$$\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}^-} = \Phi\left(\frac{\ln\left(\frac{\epsilon(m-\gamma)}{m\gamma(m-\epsilon)} - \frac{1-m}{m}\right) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}\right)$$

□

It is clear that the extreme situations occur in the same cases analyzed previously. In the case of $m \leq 1$ and $\gamma > 0$:

$$\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}^-} = 0$$

In the case of $\gamma < 0$:

$$\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}^-} = 1$$

Local cash-lock probability Considering $\epsilon \geq 0$, $\gamma \leq 0$ and $m > 1$, the γ-ϵ *cash-lock probability* in random floor CPPI for the period $(t_i, t_{i+1}]$ with $i = 0, 1, 2, \ldots, n-1$ is given by

$$\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} = \Phi\left(\frac{\ln\left(\frac{B(t_i)m(\epsilon-\gamma)}{\zeta(t_i)(m(1-c)-\epsilon)(m-\gamma)}\right) - (\mu_I - r - \frac{\sigma_I^2}{2})\frac{T}{n}}{\sigma_I\sqrt{\frac{T}{n}}}\right) \tag{3.73}$$

indeed for $\gamma > 0$ we have to use (as before) the approximation method of Fenton and Wilkinson and we arrive to the following result:

$$\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} \cong \Phi\left(\frac{\ln\left(\frac{d(t_i)\sqrt{h(a(t_i),\bar{b}(t_i))}}{f^2(a(t_i),\bar{b}(t_i))}\right)}{\sqrt{\ln\left(\frac{h(a(t_i),\bar{b}(t_i))}{f^2(a(t_i),\bar{b}(t_i))}\right)}}\right) \tag{3.74}$$

where

$$\bar{b}(t_i) = (1-c)b(t_i)$$

$\forall\ i = 0, 1, 2, \ldots, n-1$.
$a(t_i)$, $b(t_i)$, $d(t_i)$, f and h are defined previously in paragraph 3.3.4 for NPV floor.

Setting $\epsilon = 0$ in both cases we obtain the *cash-lock probability*.
The cash-lock probability is 0 in the case of $\gamma > 0$ if we have $0 \leq m \leq 1$.

Proof. First, we calculate the the γ-ϵ *cash-lock probability* for $\gamma \leq 0$, in this case we have that the cushion in pre-payment instant t_i is negative or zero $C(t_i \leq 0)$. We have the following post-payment dynamics as a result of (3.60) and (3.62) (looking at $C(t_i \leq 0)$):

$$Y(t_{i+1}) = Y(t_i)e^{r(t_{i+1}-t_i)} + \zeta(t_{i+1})$$
$$C(t_{i+1}) = C(t_i)e^{r(t_{i+1}-t_i)} + (1-c)\zeta(t_{i+1})$$

Considering the time step constant $(t_{i+1} - t_i = \frac{T}{n})$ the probability is given as:

$$\begin{aligned}
\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} &= \mathbb{P}\left(\frac{mC(t_i)e^{r\frac{T}{n}} + m(1-c)\zeta(t_{i+1})}{Y(t_i)e^{r\frac{T}{n}} + \zeta(t_{i+1})} \leq \epsilon \middle| C(t_i) = \frac{\gamma B(t_i)}{m-\epsilon}\right) \\
&= \mathbb{P}\left((C(t_i)(m-\epsilon) - \epsilon B(t_i))e^{r\frac{T}{n}} \leq (\epsilon - m(1-c))\zeta(t_{i+1}) \middle| C(t_i = \frac{\gamma B(t_i)}{m-\gamma})\right) \\
&\textit{using } (3.2) \\
&= \mathbb{P}\left((m(1-c)-\epsilon)\zeta I(t_i)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I(W(t_{i+1})-W(t_i))} \leq \left(\epsilon - \frac{\gamma(m-\epsilon)}{m-\gamma}\right)B(t_i)e^{r\frac{T}{n}}\right) \\
&\textit{taking logarithm of both side} \\
&= \mathbb{P}\left((\mu_I - \frac{\sigma_I^2}{2})\frac{T}{n} + \sigma_I(W(t_{i+1}) - W(t_i)) \leq \ln\left(\frac{\left(\epsilon - \frac{\gamma(m-\epsilon)}{m-\gamma}\right)B(t_i)e^{r\frac{T}{n}}}{(m(1-c)-\epsilon)\zeta(t_i)}\right)\right) \\
&= \mathbb{P}\left(W(t_{i+1}) - W(t_i) \leq \frac{\ln\left(\frac{m(\epsilon-\gamma)B(t_i)}{(m-\gamma)(m(1-c)-\epsilon)\zeta(t_i)}\right) - (\mu_I - r - \frac{\sigma_I^2}{2})\frac{T}{n}}{\sigma_I}\right) \\
&= \Phi\left(\frac{\ln\left(\frac{B(t_i)m(\epsilon-\gamma)}{\zeta(t_i)(m(1-c)-\epsilon)(m-\gamma)}\right) - (\mu_I - r - \frac{\sigma_I^2}{2})\frac{T}{n}}{\sigma_I\sqrt{\frac{T}{n}}}\right)
\end{aligned} \tag{3.75}$$

which completes the first part of the proof.

Now, we find the γ-ϵ *cash-lock probability* for $\gamma > 0$.
The post-payment dynamics follow from (3.60) and (3.62) (looking at $C(t_i > 0)$):

$$Y(t_{i+1}) = mC(t_i)\frac{S_{t_{i+1}}}{S_{t_i}} + (Y(t_i) - mC(t_i))e^{r(t_{i+1}-t_i)} + \zeta(t_{i+1})$$

$$C(t_{i+1}) = C(t_i)\left(m\frac{S_{t_{i+1}}}{S_{t_i}} + (1-m)e^{r(t_{i+1}-t_i)}\right) + (1-c)\zeta(t_{i+1})$$

Substituting these relations into the definition (3.64), we obtain:

$$\begin{aligned}
\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} &= \mathbb{P}\left(\frac{mC(t_i)\left(m\frac{S(t_{i+1})}{S(t_i)} + (1-m)e^{r\frac{T}{n}}\right) + (1-c)\zeta(t_{i+1})}{mC(t_i)\frac{S(t_{i+1})}{S(t_i)} + (Y(t_i) - mC(t_i))e^{r\frac{T}{n}} + \zeta(t_{i+1})} \leq \epsilon \middle| C(t_i) = \frac{\gamma B(t_i)}{m-\gamma}\right) \\
&= \mathbb{P}\Bigg(C(t_i)\left(m(m-\epsilon)\frac{S(t_i)}{S(t_i)} + (m-\epsilon)(1-m)e^{r\frac{T}{n}}\right) \\
&\qquad \leq (\epsilon - m(1-c))\zeta(t_{i+1}) + \epsilon B(t_i)e^{r\frac{T}{n}} \Bigg| C(t_i) = \frac{\gamma B(t_i)}{m-\gamma}\Bigg) \\
&= \mathbb{P}\Bigg(\frac{\gamma B(t_i)}{m-\gamma}\left(m(m-\epsilon)\frac{S(t_i)}{S(t_i)} + (m-\epsilon)(1-m)e^{r\frac{T}{n}}\right) - \epsilon B(t_i)e^{r\frac{T}{n}} \\
&\qquad \leq (\epsilon - m(1-c))\zeta(t_{i+1})\Bigg) \\
&= \mathbb{P}\Bigg(\frac{\gamma B(t_i)}{m-\gamma}\left(m(m-\epsilon)\frac{S(t_i)}{S(t_i)} + (m-\epsilon)(1-m)e^{r\frac{T}{n}}\right) - \epsilon B(t_i)e^{r\frac{T}{n}} \\
&\qquad \leq (\epsilon - m(1-c))\zeta I(t_i)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I(W(t_{i+1}) - W(t_i))}\Bigg) \\
&= \mathbb{P}\Bigg(\frac{\gamma m(m-\epsilon)B(t_i)}{m-\gamma}e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n} + \sigma_S(W(t_{i+1}) - W(t_i))} + \\
&\qquad + (m(1-c) - \epsilon)\zeta I(t_i)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I(W(t_{i+1}) - W(t_i))} \\
&\qquad \leq B(t_i)e^{r\frac{T}{n}}\left(\epsilon - \frac{(1-m)(m-\epsilon)\gamma}{m-\gamma}\right)\Bigg)
\end{aligned} \tag{3.76}$$

We denote the random variable in the probability as:

$$x_1 = e^{\sigma_S(W(t_{i+1}) - W(t_i))}$$

$$x_2 = e^{\sigma_I(W(t_{i+1}) - W(t_i))}$$

At this point, knowing $W(t_{i+1}) - W(t_i) \sim N(0, \frac{T}{n})$, it is clear that x_1 and x_2 have log-normal distribution:

$$x_1 \sim LogN(0, \sigma_S^2 \frac{T}{n})$$

$$x_2 \sim LogN(0, \sigma_I^2 \frac{T}{n})$$

Using the following notation

$$a(t_i) = B(t_i)\gamma m \frac{(m-\epsilon)}{m-\gamma} e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}$$

$$\bar{b}(t_i) = (1-c)b(t_i)$$

$$d(t_i) = B(t_i)e^{r\frac{T}{n}}\left(\epsilon - \frac{(1-m)(m-\epsilon)\gamma}{m-\gamma}\right)$$

where $b(t_i) = \zeta(t_i)(m-\epsilon)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n}}$ the searched probability (3.76) takes the form

$$\mathbb{P}\big(a(t_i)x_1 + \bar{b}(t_i)x_2 \leq d(t_i)\big)$$

We have the probability density function of a sum of two lognormal random variables, assuming $X = a(t_i)x_1 + \bar{b}(t_i)x_2$, the FW technique asserts that:

$$X = e^Y, \quad where\ Y \sim N(\mu_X, \sigma_X^2)$$

As a consequence $X \sim LogN(\mu_X, \sigma_X^2)$, where the mean and the variance of this approximated distribution are:

$$\mu_X = \ln\left(\frac{\left(a(t_i)e^{\frac{1}{2}\sigma_S^2\frac{T}{n}} + \bar{b}(t_i)e^{\frac{1}{2}\sigma_I^2\frac{T}{n}}\right)^2}{\left(a(t_i)^2e^{2\sigma_S^2\frac{T}{n}} + \bar{b}(t_i)^2e^{2\sigma_I^2\frac{T}{n}} + 2a(t_i)\bar{b}(t_i)e^{\frac{1}{2}(\sigma_S^2+\sigma_I^2)\frac{T}{n}}\right)^{\frac{1}{2}}}\right)$$

$$\sigma_X^2 = \ln\left(\frac{a(t_i)^2e^{2\sigma_S^2\frac{T}{n}} + \bar{b}(t_i)^2e^{2\sigma_I^2\frac{T}{n}} + 2a(t_i)\bar{b}(t_i)e^{\frac{1}{2}(\sigma_S^2+\sigma_I^2)\frac{T}{n}}}{\left(a(t_i)e^{\frac{1}{2}\sigma_S^2\frac{T}{n}} + \bar{b}(t_i)e^{\frac{1}{2}\sigma_I^2\frac{T}{n}}\right)^2}\right)$$

We can now approximate the γ-ϵ *cash-lock probability* for $\gamma > 0$ as follow:

$$\begin{aligned}
\mathbb{P}\big((a(t_i)x_1 + \bar{b}(t_i)x_2 \leq d(t_i)\big) &\cong \mathbb{P}\big(X \leq d(t_i)\big) \\
&= \mathbb{P}\big(e^Y \leq d(t_i)\big) \\
&= \mathbb{P}\big(Y \leq \ln\left(d(t_i)\right)\big) \\
&= \mathbb{P}\left(\frac{Y - \mu_X}{\sigma_X} \leq \frac{\ln\left(d(t_i)\right) - \mu_X}{\sigma_X}\right) \\
&= \Phi\left(\frac{\ln\left(d(t_i)\right) - \mu_X}{\sigma_X}\right)
\end{aligned}$$

By introducing the notation

$$f(a(t_i), \bar{b}(t_i)) = a(t_i)e^{\frac{1}{2}\sigma_S^2\frac{T}{n}} + \bar{b}(t_i)e^{\frac{1}{2}\sigma_I^2\frac{T}{n}}$$

$$h(a(t_i),\bar{b}(t_i)) = a(t_i)^2 e^{2\sigma_S^2 \frac{T}{n}} + \bar{b}(t_i)^2 e^{2\sigma_I^2 \frac{T}{n}} + 2a(t_i)\bar{b}(t_i)e^{\frac{1}{2}(\sigma_S^2+\sigma_I^2)\frac{T}{n}}$$

we found

$$\bar{\mathbb{P}}^{CL_{\gamma,\epsilon}}_{t_i,t_{i+1}} \cong \Phi\left(\frac{\ln\left(\frac{d(t_i)\sqrt{h(a(t_i),\bar{b}(t_i))}}{f^2(a(t_i),\bar{b}(t_i))}\right)}{\sqrt{\ln\left(\frac{h(a(t_i),\bar{b}(t_i))}{f^2(a(t_i),\bar{b}(t_i))}\right)}}\right)$$

which concludes the proof.

□

Gap risk

This section examines the gap risk, which may be thought of as a subset of the cash-lock risk. A gap develops when the overall wealth at the time of rebalancing falls below the floor. The gap is then simply the difference between the floor and the wealth. It is the consequence of a significant loss in risky asset investment between rebalancing times, and it cannot be avoided in the discrete-time context with a multiplier $m > 1$. As a result, if a gap emerges, the CPPI portfolio will be unable to provide the guaranteed capital protection. The greater the value of the multiplier m, the greater the gap danger. The value $\frac{1}{m}$ is also known as the gap size, which refers to the greatest loss that may be absorbed between two rebalancing dates before the portfolio value collapses through the floor.

We examine the risk measures *shortfall probability* and *expected shortfall* to estimate the efficiency of each CPPI method outlined above.

Shortfall probability The probability of the final wealth being less than the guaranteed amount is referred to as the shortfall probability. We denote with $\mathbb{P}^{SF}$ the shortfall probability, that is probability that $Y(T)$ is less than the guaranteed amount $B(T)$. We know that the inequality $Y(T) > B(T)$ is identical to $C(T) < 0$; consequently, $\mathbb{P}^{SF}$ may be defined as follows:

$$\mathbb{P}^{SF} = \mathbb{P}\big(C(T) < 0\big) \tag{3.77}$$

The probability that the cushion is negative after one time step, given the cushion is non-negative before, it is the local shortfall probability $\mathbb{P}^{LSF}$, and it is defined as:

$$\mathbb{P}^{LSF}_{t_i,t_{i+1}} = \mathbb{P}\big(C(t_{i+1}) < 0 \big| C(t_i) > 0\big) \tag{3.78}$$

$\forall i = 0, 1, 2, \ldots, n-1$.

It is worth noting that the definition (3.78) is a subset of the cash-lock probability (3.65) for $\epsilon = 0$ and $\gamma > 0$. As a result, when a shortfall develops, the portfolio immediately enters cash-lock mode since the complete wealth is invested in the cash asset until the cushion turns positive again. It is also important to take into account that the local shortfall probability for the period (t_i, t_{i+1}) only considers the cushion value at the conclusion of the period. Temporary under-performance followed by recovery until t_{i+1} is hence not seen as an issue. Thus, in general, we do the analysis for two time instants: at time t_{i+1}, when the contribution payment has been made, and at a time before time t_{i+1}, when the payment has not yet arrived. We denote this latter probability as

$$\mathbb{P}^{LSF}_{t_i,t^-_{i+1}} = \mathbb{P}\big(C(t^-_{i+1}) < 0 \big| C(t_i) > 0\big) \tag{3.79}$$

$\forall i = 0, 1, 2, \ldots, n-1.$

Expected shortfall The expected shortfall, on the other hand, measures the amount that is lost if a shortfall occurs. E^{SF}, with which we denoted the expected shortfall, is the amount which is lost if a shortfall occurs, that is:

$$E^{SF} = \mathbb{E}\big(-C(T) \big| C(T) < 0\big) \tag{3.80}$$

Considering the path-dependent structure of the proposed strategies, localized version of E^{SF} is defined and denoted by E^{LSF}. The local expected shortfall at time t_{i+1} is given as

$$E^{LSF}_{t^-_{i+1}} = \mathbb{E}\big(-C(t^-_{i+1}) \big| C(t^-_{i+1}) < 0\big) \tag{3.81}$$

We now define the above probability measures in the case of CPPI strategy with NPV floor.

Gap risk in NPV floor In the CPPI strategy with NPV floor we first examine the local shortfall probability before and then after the contribution is received; from this last measure, we can obtain the probability of shortfall as the chances of experiencing at least one shortfall within the time period studied.
Later, in the same way, we expose the expected shortfall in t^-_{i+1} and in t_{i+1}, in each instants, taking into account both the scenario when the cushion value is larger than zero and the case where it is not positive $\forall\ i = 0, 1, 2, \ldots, n-1$.

Shortfall probability First we find the local shortfall probability at time t^-_{i+1}, when payment has not yet arrived. The local shortfall probability at time t^-_{i+1} in NPV floor CPPI is given by:

$$\mathbb{P}^{LSF}_{t_i,t^-_{i+1}} = \Phi\left(\frac{\ln\left(\frac{m-1}{m}\right) - \left(\mu_S - r - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}\right) \tag{3.82}$$

$\forall\ i = 0, 1, 2, \ldots, n-1.$

After the payment has been made, at time t_{i+1}, the local shortfall probability can be approximated as follow:

$$\mathbb{P}^{LSF}_{t_i,t_{i+1}} \cong \Phi\left(\frac{\ln\left(\frac{e^{r\frac{T}{n}}\left(\frac{m-1}{m}\right)\sqrt{h(\widetilde{a}(t_i),\widetilde{b}(t_i))}}{f^2(\widetilde{a}(t_i),\widetilde{b}(t_i))}\right)}{\sqrt{\ln\left(\frac{h(\widetilde{a}(t_i),\widetilde{b}(t_i))}{f^2(\widetilde{a}(t_i),\widetilde{b}(t_i))}\right)}}\right) \tag{3.83}$$

$\forall\ i = 0, 1, 2, \dots, n-1.$
Where

$$\widetilde{a}(t_i) = e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}$$

and

$$\widetilde{b}(t_i) = e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}} \frac{\zeta(t_i)}{mC(t_i)}$$

the functions f and h are the same as those given for the probability of cash lock in (3.69); Φ is the cumulative distribution function of standard normal distribution.

Proof. In the first part of the proof we calculated the probability at time t_{i+1}^-:

$$\begin{aligned}
\mathbb{P}^{LSF}_{t_i,t_{i+1}^-} &= \mathbb{P}(C(t_{i+1}^-) < 0 | C(t_i) > 0) \\
&= \mathbb{P}\Big(C(t_i)\Big(m\frac{S_{t_{i+1}}}{S_{t_i}} + (1-m)e^{r(t_{i+1}-t_i)}\Big) < 0 | C(t_i) > 0\Big) \\
& \textit{we can divide on both side by } C(t_i) \textit{ since } C(t_i) > 0 \\
& \textit{we consider time step constant}: \ t_{i+1} - t_i = \frac{T}{n} \\
&= \mathbb{P}\Big(\frac{S_{t_{i+1}}}{S_{t_i}} < \frac{m-1}{m}e^{r\frac{T}{n}}\Big) \\
&= \mathbb{P}\Big(e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n} + \sigma_S\left(W(t_{i+1}) - W(t_i)\right)} < \frac{m-1}{m}e^{r\frac{T}{n}}\Big) \\
& \textit{we take logarithm of both side} \\
&= \mathbb{P}\Big((\mu_S - \frac{\sigma_S^2}{2})\frac{T}{n} + \sigma_S(W(t_{i+1}) - W(t_i)) < \ln(\frac{m-1}{m}) + r\frac{T}{n}\Big) \\
&= \mathbb{P}\Big(W(t_{i+1}) - W(t_i) < \frac{1}{\sigma_S}\Big(\ln(\frac{m-1}{m}) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}\Big)\Big) \\
&= \mathbb{P}\Big(\frac{W(t_{i+1}) - W(t_i)}{\sqrt{\frac{T}{n}}} < \frac{1}{\sigma_S\sqrt{\frac{T}{n}}}\Big(\ln(\frac{m-1}{m}) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}\Big)\Big) \\
&= \Phi\Big(\frac{\ln(\frac{m-1}{m}) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}\Big)
\end{aligned}$$

The after-payment local shortfall probability (at time t_{i+1}) is given by

$$\begin{aligned}
\mathbb{P}^{LSF}_{t_i,t_{i+1}} &= \mathbb{P}(C(t_{i+1}) < 0 \big| C(t_i) > 0) \\
&= \mathbb{P}\Big(C(t_i)\Big(m\frac{S_{t_{i+1}}}{S_{t_i}} + (1-m)e^{r(t_{i+1}-t_i)}\Big) + \zeta(t_{i+1}) < 0 \Big| C(t_i) > 0 \Big) \\
& \textit{we can divide on both side by } C(t_i) \textit{ since } C(t_i) > 0 \\
& \textit{we consider time step constant}: \ t_{i+1} - t_i = \frac{T}{n} \\
&= \mathbb{P}\Big(\frac{S_{t_{i+1}}}{S_{t_i}} + \frac{\zeta(t_{i+1})}{mC(t_i)} < \frac{m-1}{m} e^{r\frac{T}{n}} \Big| C(t_i) > 0 \Big) \\
& \textit{using } \zeta(t_{i+1}) = \zeta I(t_{i+1}) \\
&= \mathbb{P}\Big(e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n} + \sigma_S\left(W(t_{i+1}) - W(t_i)\right)} \\
&\qquad + \frac{\zeta(t_i)}{mC(t_i)} e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I\left(W(t_{i+1}) - W(t_i)\right)} < \frac{m-1}{m} e^{r\frac{T}{n}} \Big)
\end{aligned}$$

We have, again, a sum of two log-normal random variables:

$$x_1 = e^{\sigma_S\left(W(t_{i+1}) - W(t_i)\right)} \qquad x_1 \sim LogN(0, \sigma_S^2 \frac{T}{n})$$

and

$$x_2 = e^{\sigma_I\left(W(t_{i+1}) - W(t_i)\right)} \qquad x_1 \sim LogN(0, \sigma_I^2 \frac{T}{n})$$

Setting

$$\widetilde{a}(t_i) = e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}$$

and

$$\widetilde{b}(t_i) = e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}} \frac{\zeta(t_i)}{mC(t_i)}$$

the searched probability becomes

$$\mathbb{P}\Big(\widetilde{a}(t_i)x_1 + \widetilde{b}(t_i)x_2 < \frac{m-1}{m} e^{r\frac{T}{n}} \Big)$$

Using the Fenton-Wilkinson method, we can assert that the random variable $X = \widetilde{a}(t_i)x_1 + \widetilde{b}(t_i)x_2$ can be approximated by a log-normal distribution. $X = e^Y$, where Y is normally distributed and have mean and variance equal to:

$$\mu_X = \ln\left(\frac{f^2(\widetilde{a}(t_i), \widetilde{b}(t_i))}{\sqrt{h(\widetilde{a}(t_i), \widetilde{b}(t_i))}} \right)$$

$$\sigma_X^2 = \ln\left(\frac{h(\widetilde{a}(t_i), \widetilde{b}(t_i))}{f^2(\widetilde{a}(t_i), \widetilde{b}(t_i))} \right)$$

where f and h are defined in (3.69).

The probability in therefore calculated as follow

$$\begin{aligned}\mathbb{P}\Big(\widetilde{a}(t_i)x_1+\widetilde{b}(t_i)x_2<\frac{m-1}{m}e^{r\frac{T}{n}}\Big) &= \mathbb{P}\Big(\ln\big(\widetilde{a}(t_i)x_1+\widetilde{b}(t_i)x_2\big)<\ln\big(\frac{m-1}{m}e^{r\frac{T}{n}}\big)\Big)\\ &= \mathbb{P}\Big(\frac{\ln\big(\widetilde{a}(t_i)x_1+\widetilde{b}(t_i)x_2\big)-\mu_X}{\sigma_X^2}<\frac{\ln\big(\frac{m-1}{m}e^{r\frac{T}{n}}\big)-\mu_X}{\sigma_X^2}\Big)\\ &\cong \Phi\Bigg(\frac{\ln\big(\frac{m-1}{m}e^{r\frac{T}{n}}\big)-\mu_X}{\sigma_X^2}\Bigg)\\ &= \Phi\Bigg(\frac{\ln\Bigg(\frac{e^{r\frac{T}{n}}\left(\frac{m-1}{m}\right)\sqrt{h\left(\widetilde{a}(t_i),\widetilde{b}(t_i)\right)}}{f^2\left(\widetilde{a}(t_i),\widetilde{b}(t_i)\right)}\Bigg)}{\ln\Bigg(\frac{h\left(\widetilde{a}(t_i),\widetilde{b}(t_i)\right)}{f^2\left(\widetilde{a}(t_i),\widetilde{b}(t_i)\right)}\Bigg)}\Bigg)\end{aligned}$$

□

Following the same logic used in the section 2.5.3 we obtain the shortfall probability:

$$\mathbb{P}^{SF}=1-\prod_{i=0}^{n-1}\left(1-\mathbb{P}^{LSF}_{t_i,t_{i+1}}\right) \tag{3.84}$$

Expected shortfall The local expected shortfall at time t_{i+1}^- for CPPI strategy with NPV floor is given by

$$E^{LSF}_{t_{i+1}^-}=\begin{cases}\dfrac{-C(t_i)F_1}{\mathbb{P}^{LSF}_{t_i,t_{i+1}^-}} & C(t_i)>0\\ -C(t_i)e^{r\frac{T}{n}} & C(t_i)\le 0\end{cases} \tag{3.85}$$

At the end of each period payment, we can approximate E^{LSF} as follows:

$$E^{LSF}_{t_{i+1}}\cong\begin{cases}\dfrac{-C(t_i)F_2-\zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi\Big(\widetilde{d}(t_i)-\sigma_I\sqrt{\frac{T}{n}}\Big)}{\mathbb{P}^{LSF}_{t_i,t_{i+1}}} & C(t_i)>0\\ -C(t_i)e^{r\frac{T}{n}}-\dfrac{\zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi\Big(\widehat{d}(t_i)-\sigma_I\sqrt{\frac{T}{n}}\Big)}{\Phi(\widehat{d})} & C(t_i)\le 0\end{cases} \tag{3.86}$$

$\forall\ i=0,1,2,\ldots,n-1$.
Where

$$F_1=me^{\mu_S\frac{T}{n}}\Phi\Big(d-\sigma_S\sqrt{\frac{T}{n}}\Big)+(1-m)e^{r\frac{T}{n}}\mathbb{P}^{LSF}_{t_i,t_{i+1}^-}$$

$$F_2=me^{\mu_S\frac{T}{n}}\Phi\Big(\widetilde{d}(t_i)-\sigma_S\sqrt{\frac{T}{n}}\Big)+(1-m)e^{r\frac{T}{n}}\mathbb{P}^{LSF}_{t_i,t_{i+1}}$$

with

$$d = \frac{\ln\left(\frac{m-1}{m}\right) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}$$

$$\widehat{d}(t_i) = \frac{\ln\left(-\frac{C(t_i)}{\zeta(t_i)}\right) - (\mu_I - r - \frac{\sigma_L^2}{2})\frac{T}{n}}{\sigma_I\sqrt{\frac{T}{n}}}$$

$$\widetilde{d}(t_i) = \frac{\ln\left(\frac{e^{r\frac{T}{n}}\left(\frac{m-1}{m}\right)\sqrt{h(\widetilde{a}(t_i),\widetilde{b}(t_i))}}{f^2(\widetilde{a}(t_i),\widetilde{b}(t_i))}\right)}{\sqrt{\ln\left(\frac{h(\widetilde{a}(t_i),\widetilde{b}(t_i))}{f^2(\widetilde{a}(t_i),\widetilde{b}(t_i))}\right)}}$$

$\widetilde{a}(t_i)$ and $\widetilde{b}(t_i)$ are given in (3.83). Φ is always the cumulative distribution function of standard normal distribution.

Proof. In the first section of the proof we use the following notation for the shortfall event during the period (t_{i-1}, t_i)

$$A_i = \left\{\frac{S(t_i)}{S(t_{i-1})} < \left(\frac{m-1}{m}\right)e^{r\frac{T}{n}}\right\}$$

and its complement event is defined as

$$A_i^c = \left\{\frac{S(t_i)}{S(t_{i-1})} \leq \left(\frac{m-1}{m}\right)e^{r\frac{T}{n}}\right\}$$

that is the event of shortfall probability during the period (t_{i-1}, t_i) is zero.

In the first step of proof the pre-payment $E^{LSF}_{t^-_{i+1}}$ will be calculated for $C(t_i) > 0$.

$$\begin{aligned}
E^{LSF}_{t^-_{i+1}} &= \mathbb{E}\big(-C(t^-_{i+1})\big|C(t^-_{i+1}) < 0\big) \\
&= \frac{\mathbb{E}\big(-C(t^-_{i+1})\mathbf{1}_{A_{i+1}}\big)}{\mathbb{P}(A_{i+1})} \\
&= -C(t_i^-)\frac{\mathbb{E}\Big((m\frac{S(t_{i+1})}{S(t_i)} + (1-m)e^{r(t_{i+1}-t_i)})\mathbf{1}_{A_{i+1}}\Big)}{\mathbb{P}(A_{i+1})}
\end{aligned}$$

Considering each time step $(t_{i+1} - t_i)$ equal to $\frac{T}{n}$ we compute the expectation in the numerator as follows:

$$\begin{aligned}
&\mathbb{E}\Big((m\frac{S(t_{i+1})}{S(t_i)} + (1-m)e^{r(t_{i+1}-t_i)})\mathbf{1}_{A_{i+1}}\Big) \\
&= m\mathbb{E}\Big(\frac{S(t_{i+1})}{S(t_i)}\mathbf{1}_{A_{i+1}}\Big) + (1-m)e^{r\frac{T}{n}}\mathbb{E}\Big(\mathbf{1}_{A_{i+1}}\Big) \\
&= me^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}\mathbb{E}\big(e^{\sigma_S(W(t_{i+1})-W(t_i))}\mathbf{1}_{A_{i+1}}\big) + (1-m)e^{r\frac{T}{n}}\mathbb{E}\Big(\mathbf{1}_{A_{i+1}}\Big)
\end{aligned}$$

Now, we must calculate the two expectations.
The last one is the local shortfall probability in an instant before payment:

$$\mathbb{E}\Big(\mathbf{1}_{A_{i+1}}\Big) = \mathbb{P}^{LSF}_{t_i,t^-_{i+1}}$$

as can be clearly seen from the proof of the equation (3.82).

For the second expectation, we consider $W(t_{i+1}) - W(t_i) \sim N(0, \frac{T}{n})$. Therefore we can write $\sigma_S(W(t_{i+1}) - W(t_i)) = \sqrt{\frac{T}{n}}Z$ where $Z \sim N(0,1)$.

$$\begin{aligned}
\mathbb{E}\big(e^{\sigma_S(W(t_{i+1})-W(t_i))}\mathbf{1}_{A_{i+1}}\big) &= \mathbb{E}\big(e^{\sigma_S\sqrt{\frac{T}{n}}Z}\mathbf{1}_{A_{i+1}}\big)\\
&= \int_{\mathbf{1}_{A_{i+1}}} e^{\sigma_S\sqrt{\frac{T}{n}}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\\
&= \int_{-\infty}^{d} e^{\sigma_S\sqrt{\frac{T}{n}}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\\
&= \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}+x\sigma_S\sqrt{\frac{T}{n}}}dx\\
&\quad \textit{we add and subtract the quantity } \frac{\sigma_S^2\frac{T}{n}}{2} \textit{ in the exponential function}\\
&= \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}+x\sigma_S\sqrt{\frac{T}{n}}-\frac{\sigma_S^2\frac{T}{n}}{2}+\frac{\sigma_S^2\frac{T}{n}}{2}}dx\\
&= \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(x^2-2x\sigma_S\sqrt{\frac{T}{n}}+\sigma_S^2\frac{T}{n}\right)+\frac{\sigma_S^2\frac{T}{n}}{2}}dx\\
&= \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(x^2-2x\sigma_S\sqrt{\frac{T}{n}}+\sigma_S^2\frac{T}{n}\right)+\frac{\sigma_S^2\frac{T}{n}}{2}}dx\\
&= \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}}e^{\frac{\sigma_S^2\frac{T}{n}}{2}}e^{-\frac{1}{2}\left(x^2-2x\sigma_S\sqrt{\frac{T}{n}}+\sigma_S^2\frac{T}{n}\right)}dx\\
&= e^{\frac{\sigma_S^2}{2}\frac{T}{n}}\int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(x-\sigma_S\sqrt{\frac{T}{n}}\right)^2}dx
\end{aligned}$$

within the integral we have the probability density function of a normal distribution with mean equal to $\sigma_S\sqrt{\frac{T}{n}}$ and standard deviation equal to 1, then

$$\mathbb{E}\big(e^{\sigma_S(W(t_{i+1})-W(t_i))}\mathbf{1}_{A_{i+1}}\big) = e^{\frac{\sigma_S^2}{2}\frac{T}{n}}\Phi\left(d - \sigma_S\sqrt{\frac{T}{n}}\right)$$

Hence we have

$$\begin{aligned} F_1 &= \mathbb{E}\Big((m\frac{S(t_{i+1})}{S(t_i)} + (1-m)e^{r(t_{i+1}-t_i)})\mathbf{1}_{A_{i+1}} \Big) \\ &= me^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}} e^{\frac{\sigma_S^2}{2}\frac{T}{n}} \Phi\Big(d - \sigma_S\sqrt{\frac{T}{n}} \Big) + (1-m)e^{r\frac{T}{n}} \mathbb{P}^{LSF}_{t_i, t^-_{i+1}} \\ &= me^{\mu_S \frac{T}{n}} \Phi\Big(d - \sigma_S\sqrt{\frac{T}{n}} \Big) + (1-m)e^{r\frac{T}{n}} \mathbb{P}^{LSF}_{t_i, t^-_{i+1}} \end{aligned}$$

and so

$$E^{LSF}_{t^-_{i+1}} = \frac{-C(t_i)F_1}{\mathbb{P}^{LSF}_{t_i, t^-_{i+1}}}$$

which complete the proof of the value of $E^{LSF}_{t^-_{i+1}}$ for $C(t_i) > 0$.

Now we assumes that $C(t_i) \leq 0$. In this case the value of the cushion is only invested in risk-free asset after time t_i. Therefore, it holds

$$\begin{aligned} E^{LSF}_{t^-_{i+1}} &= \mathbb{E}\big(-C(t^-_{i+1}) \big| C(t^-_{i+1}) < 0 \big) \\ &= \mathbb{E}\big(-C(t_i)e^{r\frac{T}{n}} \big) = -C(t_i)e^{r\frac{T}{n}} \end{aligned}$$

In the follows part of the proof we calculate the post-payment local expected shortfall $E^{LSF}_{t_{i+1}}$ for cases of $C(t_i)$. For this purpose, we introduce another event:

$$\begin{aligned} B_i = \Big\{ & e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n} + \sigma_S\left(W(t_{i+1} - W(t_i))\right)} + \frac{\zeta(t_i)}{mC(t_i)} e^{\left(\mu_I - \frac{\sigma_S^2}{2}\right)\frac{T}{n} + \sigma_I\left(W(t_{i+1}) - W(t_i)\right)} \\ & < \frac{(m-1)e^{r\frac{T}{n}}}{m} \Big\} \end{aligned}$$

First we analyze the case $C(t_i) > 0$.

$$E^{LSF}_{t_{i+1}} = \mathbb{E}\big(-C(t_{i+1}) \big| C(t_{i+1}) < 0 \big) = \frac{\mathbb{E}\big(-C(t_{i+1})\mathbf{1}_{B_i} \big)}{\mathbb{P}(B_i)}$$

Considering each time step $(t_{i+1} - t_i)$ equal to $\frac{T}{n}$ we compute the expectation in the numerator as follows:

$$\begin{aligned} \mathbb{E}\big(-C(t_{i+1})\mathbf{1}_{B_i} \big) =& \mathbb{E}\Bigg(-\Big(C(t_i)\Big(m\frac{S(t_{i+1})}{S(t_i)} + (1-m)e^{r(t_{i+1}-t_i)} \Big) - \zeta(t_{i+1}) \Big)\mathbf{1}_{B_i} \Bigg) \\ &= -C(t_i)\Bigg(m\mathbb{E}\Big(\frac{S(t_{i+1})}{S(t_i)}\mathbf{1}_{B_{i+1}} \Big) + (1-m)e^{r\frac{T}{n}}\mathbb{E}(\mathbf{1}_{B_i}) \Bigg) - \mathbb{E}\Big(\zeta(t_{i+1})\mathbf{1}_{B_i} \Big) \end{aligned}$$

From the proof of (3.83) it is clear that

$$\mathbb{E}(\mathbf{1}_{B_i}) = \mathbb{P}(B_i) = \mathbb{P}^{LSF}_{t_i, t_{i+1}}$$

Using comparable calculations to the previous section of the proof, it is determined that

$$\mathbb{E}\Big(\frac{S(t_{i+1})}{S(t_i)}\mathbf{1}_{B_i}\Big) = e^{\mu_S \frac{T}{n}}\Phi(\widetilde{d} - \sigma_S\sqrt{\frac{T}{n}}) \tag{3.87}$$

Now we computer the last expectation

$$\begin{aligned}
\mathbb{E}\Big(\zeta(t_{i+1})\mathbf{1}_{B_i}\Big) &= \mathbb{E}\Big(\zeta(t_i)e^{\big(\mu_I - \frac{\sigma_I^2}{2}\big)\frac{T}{n} + \sigma_I\big(W(t_{i+1}) - W(t_i)\big)}\mathbf{1}_{B_i}\Big) \\
&= \zeta(t_i)e^{\big(\mu_I - \frac{\sigma_I^2}{2}\big)\frac{T}{n}}\mathbb{E}\Big(e^{\sigma_I\big(W(t_{i+1}) - W(t_i)\big)}\mathbf{1}_{B_i}\Big)
\end{aligned}$$

we follow the same logic used for the previous proof and we obtain

$$\mathbb{E}\Big(e^{\sigma_I\big(W(t_{i+1}) - W(t_i)\big)}\mathbf{1}_{B_i}\Big) = e^{\frac{\sigma_I^2}{2}\frac{T}{n}}\Phi(\widetilde{d} - \sigma_S\sqrt{\frac{T}{n}}) \tag{3.88}$$

then

$$\mathbb{E}\Big(\zeta(t_{i+1})\mathbf{1}_{B_i}\Big) = \zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi(\widetilde{d} - \sigma_I\sqrt{\frac{T}{n}})$$

Hence, using the previous assumptions, and setting

$$F_2 = me^{\mu_S\frac{T}{n}}\Phi\Big(\widetilde{d}(t_i) - \sigma_S\sqrt{\frac{T}{n}}\Big) + (1-m)e^{r\frac{T}{n}}\mathbb{P}^{LSF}_{t_i,t_{i+1}}$$

we have proved that the local expected shortfall for $C(t_i) > 0$ is equal to

$$\mathbb{E}^{LSF}_{t_{i+1}} = \frac{-C(t_i)F_2 - \zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi\Big(\widetilde{d}(t_i) - \sigma_I\sqrt{\frac{T}{n}}\Big)}{\mathbb{P}^{LSF}_{t_i,t_{i+1}}}$$

Finally for the proof of $\mathbb{E}^{LSF}_{t_{i+1}} = \mathbb{E}\big(-C(t_{i+1})\big|C(t_{i+1}) < 0\big)$ in the case which $C(t_i) \leq 0$ we introduce another event

$$E_i = \Big\{C(t_i)e^{r\frac{T}{n}} + \zeta(t_{i+1}) < 0\Big\}$$

Hence

$$\begin{aligned}
\mathbb{E}^{LSF}_{t_{i+1}} &= \frac{\mathbb{E}\Big(\big(-C(t_i)e^{r\frac{T}{n}} - \zeta(t_{i+1})\big)\mathbf{1}_{E_i}\Big)}{\mathbb{P}(E_i)} \\
&= \frac{-C(t_i)e^{r\frac{T}{n}} - \mathbb{E}\Big(\zeta(t_{i+1})\mathbf{1}_{E_i}\Big)}{\mathbb{P}(E_i)}
\end{aligned}$$

In this case, where $C(t_i) \leq 0$, the probability in the denominator is equal to:

$$\begin{aligned}
\mathbb{P}(E_i) &= \mathbb{P}\Big(C(t_i)e^{r\frac{T}{n}} + \zeta(t_{i+1}) < 0\Big) \\
&= \mathbb{P}\Big(C(t_i)e^{r\frac{T}{n}} + \zeta(t_i)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I\left(W(t_{i+1}) - W(t_i)\right)} < 0\Big) \\
&= \mathbb{P}\Big(e^{\sigma_I\left(W(t_{i+1}) - W(t_i)\right)} < -\frac{C(t_i)}{\zeta(t_i)}e^{r\frac{T}{n}}e^{-\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n}}\Big) \\
&\textit{taking logarithm} \\
&= \mathbb{P}\Big(\sigma_I(W(t_{i+1}) - W(t_i)) < \ln\big(-\frac{C(t_i)}{\zeta(t_i)}e^{r\frac{T}{n}}e^{-\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n}}\big)\Big) \\
&= \mathbb{P}\Big(\frac{(W(t_{i+1}) - W(t_i))}{\sqrt{\frac{T}{n}}} < \frac{1}{\sigma_I\sqrt{\frac{T}{n}}}\Big(\ln\Big(-\frac{C(t_i)}{\zeta(t_i)}\Big) - (\mu_I - \frac{\sigma_I^2}{2} - r)\frac{T}{n}\Big)\Big) \\
&= \Phi\left(\frac{\ln\Big(-\frac{C(t_i)}{\zeta(t_i)}\Big) - (\mu_I - \frac{\sigma_I^2}{2} - r)\frac{T}{n}}{\sigma_I\sqrt{\frac{T}{n}}}\right) = \Phi(\widehat{d}(t_i))
\end{aligned}$$

and in the same way that we compute $\mathbb{E}\Big(\zeta(t_{i+1})\mathbf{1}_{B_i}\Big)$ it holds that

$$\mathbb{E}\Big(\zeta(t_{i+1})\mathbf{1}_{E_i}\Big) = \zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi(\widehat{d}(t_i) - \sigma_I\sqrt{\frac{T}{n}})$$

Finally

$$\mathbb{E}^{LSF}_{t_{i+1}} = -C(t_i)e^{r\frac{T}{n}} - \frac{\zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi\Big(\widehat{d}(t_i) - \sigma_I\sqrt{\frac{T}{n}}\Big)}{\Phi(\widehat{d}(t_i))}$$

□

Gap risk in random floor In the CPPI strategy with random floor we proceed as in the previous case by examining first the local shortfall probability before and then after the contribution is received and later, in the same way, we expose the expected shortfall in t^-_{i+1} and in t_{i+1} taking into account both the scenario when the cushion value is larger than zero and the case where it is not positive $\forall\, i = 0, 1, 2, \ldots, n-1$.

It is important to note that, when the next payment is not taken into account, the cushion value $C(t^-_{i+1})$ has the same dynamics in both type of floor. Therefore the local shortfall probability and the local expected shortfall at time t^-_{i+1} for the random floor scenario is the same as for the NPV floor case.

Shortfall probability Hence, as just said, the local shortfall probability at time t^-_{i+1} in random floor CPPI $\bar{\mathbb{P}}^{LSF}_{t_i, t^-_{i+1}}$ is equal to (3.82)

$$\bar{\mathbb{P}}^{LSF}_{t_i, t^-_{i+1}} = \Phi\left(\frac{\ln\left(\frac{m-1}{m}\right) - (\mu_S - r - \frac{\sigma_S^2}{2})\frac{T}{n}}{\sigma_S\sqrt{\frac{T}{n}}}\right) \tag{3.89}$$

$\forall\ i = 0, 1, 2, \ldots, n-1$.

After payment at time t_{i+1} the cushion (and the wealth) dynamics change from the NPV floor. As a consequence now we approximate the local shortfall probability as follows

$$\bar{\mathbb{P}}^{LSF}_{t_i,t_{i+1}} \cong \Phi\left(\frac{\ln\left(\frac{e^{r\frac{T}{n}}\left(\frac{m-1}{m}\right)\sqrt{h(\widetilde{a}(t_i),\widehat{b}(t_i))}}{f^2(\widetilde{a}(t_i),\widehat{b}(t_i))}\right)}{\sqrt{\ln\left(\frac{h(\widetilde{a}(t_i),\widehat{b}(t_i))}{f^2(\widetilde{a}(t_i),\widehat{b}(t_i))}\right)}} \right) \tag{3.90}$$

$\forall\ i = 0, 1, 2, \ldots, n-1$ and with

$$\widehat{b}(t_i) = (1-c)\widetilde{b}(t_i)$$

Where Φ is the cumulative distribution function of standard normal random variable; f, h, $\widetilde{a}(t_i)$ and $\widetilde{b}(t_i)$ are the same defined before.

Proof. The first proof can be accomplished in the same manner as for (3.82).

For the second proof, it holds

$$\begin{aligned}
\bar{\mathbb{P}}^{LSF}_{t_i,t_{i+1}} &= \mathbb{P}(C(t_{i+1}) < 0 \big| C(t_i) > 0) \\
&= \mathbb{P}\Big(C(t_i)\Big(m\frac{S_{t_{i+1}}}{S_{t_i}} + (1-m)e^{r(t_{i+1}-t_i)}\Big) + (1-c)\zeta(t_{i+1}) < 0 \Big| C(t_i) > 0\Big) \\
&\quad \textit{we can divide on both side by } C(t_i) \textit{ since } C(t_i) > 0 \\
&\quad \textit{and considering time step constant}: \ t_{i+1} - t_i = \frac{T}{n} \\
&= \mathbb{P}\Big(\frac{S_{t_{i+1}}}{S_{t_i}} + \frac{(1-c)\zeta(t_{i+1})}{mC(t_i)} < \frac{m-1}{m}e^{r\frac{T}{n}} \Big| C(t_i) > 0\Big) \\
&\quad \textit{using } \zeta(t_{i+1}) = \zeta I(t_{i+1}) \\
&= \mathbb{P}\Big(e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n} + \sigma_S\left(W(t_{i+1}) - W(t_i)\right)} \\
&\qquad + \frac{(1-c)\zeta(t_i)}{mC(t_i)}e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I\left(W(t_{i+1}) - W(t_i)\right)} < \frac{m-1}{m}e^{r\frac{T}{n}}\Big)
\end{aligned}$$

We have, again, a sum of two log-normal random variables:

$$x_1 = e^{\sigma_S\left(W(t_{i+1}) - W(t_i)\right)} \qquad x_1 \sim LogN(0, \sigma_S^2\frac{T}{n})$$

and

$$x_2 = e^{\sigma_I\left(W(t_{i+1}) - W(t_i)\right)} \qquad x_1 \sim LogN(0, \sigma_I^2\frac{T}{n})$$

$\widetilde{a}(t_i)$ and $\widetilde{b}(t_i)$ are the same as before and now we have

$$\widehat{b}(t_i) = e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n}}\frac{(1-c)\zeta(t_i)}{mC(t_i)} = (1-c)\widetilde{b}(t_i)$$

the searched probability becomes

$$\mathbb{P}\Big(\widetilde{a}(t_i)x_1 + \widehat{b}(t_i)x_2 < \frac{m-1}{m}e^{r\frac{T}{n}}\Big)$$

Using the Fenton-Wilkinson method, we can assert that the random variable $X = \widetilde{a}(t_i)x_1 + \widehat{b}(t_i)x_2$ can be approximated by a log-normal distribution. $X = e^Y$, where Y is normally distributed and have mean and variance equal to:

$$\mu_X = \ln\left(\frac{f^2(\widetilde{a}(t_i), \widehat{b}(t_i))}{\sqrt{h(\widetilde{a}(t_i), \widehat{b}(t_i))}}\right)$$

$$\sigma_X^2 = \ln\left(\frac{h(\widetilde{a}(t_i), \widehat{b}(t_i))}{f^2(\widetilde{a}(t_i), \widehat{b}(t_i))}\right)$$

where f and h are defined in (3.69).

The probability in therefore calculated as follow

$$\begin{aligned}
\mathbb{P}\Big(\widetilde{a}(t_i)x_1 + \widehat{b}(t_i)x_2 < \frac{m-1}{m}e^{r\frac{T}{n}}\Big) &= \mathbb{P}\Big(\ln\big(\widetilde{a}(t_i)x_1 + \widehat{b}(t_i)x_2\big) < \ln\big(\frac{m-1}{m}e^{r\frac{T}{n}}\big)\Big) \\
&= \mathbb{P}\Big(\frac{\ln\big(\widetilde{a}(t_i)x_1 + \widehat{b}(t_i)x_2\big) - \mu_X}{\sigma_X^2} < \frac{\ln\big(\frac{m-1}{m}e^{r\frac{T}{n}}\big) - \mu_X}{\sigma_X^2}\Big) \\
&\cong \Phi\Big(\frac{\ln\big(\frac{m-1}{m}e^{r\frac{T}{n}}\big) - \mu_X}{\sigma_X^2}\Big)
\end{aligned}$$

So $\forall\ i = 0, 1, 2, \ldots, n-1$ we have

$$\bar{\mathbb{P}}^{LSF}_{t_i,t_{i+1}} \cong \Phi\left(\frac{\ln\left(\frac{e^{r\frac{T}{n}}\big(\frac{m-1}{m}\big)\sqrt{h(\widetilde{a}(t_i), \widehat{b}(t_i))}}{f^2(\widetilde{a}(t_i), \widehat{b}(t_i))}\right)}{\sqrt{\ln\left(\frac{h(\widetilde{a}(t_i), \widehat{b}(t_i))}{f^2(\widetilde{a}(t_i), \widehat{b}(t_i))}\right)}}\right) \tag{3.91}$$

that concludes the proof.

□

Expected shortfall Furthermore here, the local expected shortfall at pre-payments time t^-_{i+1} in random floor CPPI is equal to (3.85)

$$\bar{E}^{LSF}_{t^-_{i+1}} = \begin{cases} \dfrac{-C(t_i)F_1}{\bar{\mathbb{P}}^{LSF}_{t_i,t^-_{i+1}}} & C(t_i) > 0 \\ -C(t_i)e^{r\frac{T}{n}} & C(t_i) \le 0 \end{cases} \tag{3.92}$$

Indeed taking into account the end of period payment, the approximate local expected shortfall at time t_{i+1} is given by

$$\bar{E}^{LSF}_{t_{i+1}} \cong \begin{cases} \dfrac{-C(t_i)F_2 - (1-c)\zeta(t_i)e^{\mu_I \frac{T}{n}}\Phi\Big(\widetilde{d}(t_i) - \sigma_I\sqrt{\frac{T}{n}}\Big)}{\bar{\mathbb{P}}^{LSF}_{t_i,t_{i+1}}} & C(t_i) > 0 \\ -C(t_i)e^{r\frac{T}{n}} - \dfrac{(1-c)\zeta(t_i)e^{\mu_I \frac{T}{n}}\Phi\Big(\widehat{d}(t_i) - \sigma_I\sqrt{\frac{T}{n}}\Big)}{\Phi(\widehat{d})} & C(t_i) \leq 0 \end{cases} \tag{3.93}$$

$\forall\ i = 0, 1, 2, \ldots, n-1$.
Where

$$F_1 = me^{\mu_S \frac{T}{n}}\Phi\bigg(d - \sigma_S\sqrt{\frac{T}{n}}\bigg) + (1-m)e^{r\frac{T}{n}}\bar{\mathbb{P}}^{LSF}_{t_i,t^-_{i+1}}$$

$$F_2 = me^{\mu_S \frac{T}{n}}\Phi\bigg(\bar{d}(t_i) - \sigma_S\sqrt{\frac{T}{n}}\bigg) + (1-m)e^{r\frac{T}{n}}\bar{\mathbb{P}}^{LSF}_{t_i,t_{i+1}}$$

with

$$\bar{d}(t_i) = \frac{\ln\left(\frac{e^{r\frac{T}{n}}\left(\frac{m-1}{m}\right)\sqrt{h(\widetilde{a}(t_i),\widehat{b}(t_i))}}{f^2(\widetilde{a}(t_i),\widehat{b}(t_i))}\right)}{\sqrt{\ln\left(\frac{h(\widetilde{a}(t_i),\widehat{b}(t_i))}{f^2(\widetilde{a}(t_i),\widehat{b}(t_i))}\right)}}$$

d, $\widetilde{a}(t_i)$, $\widehat{b}(t_i)$ and the functions f and h are given previously. Φ is always the cumulative distribution function of standard normal random variable.

Proof. The proof for pre-payments local expected shortfall is is obtained in a similar way to the proof of (3.86).

In the follows part of the proof we calculate the post-payment local expected shortfall $E^{LSF}_{t_{i+1}}$ for cases of $C(t_i)$. For this purpose, we introduce the event:

$$\bar{B}_i = \bigg\{e^{\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{T}{n} + \sigma_S\left(W(t_{i+1} - W(t_i))\right)} + \frac{(1-c)\zeta(t_i)}{mC(t_i)}e^{\left(\mu_I - \frac{\sigma_S^2}{2}\right)\frac{T}{n} + \sigma_I\left(W(t_{i+1}) - W(t_i)\right)} < \frac{(m-1)e^{r\frac{T}{n}}}{m}\bigg\}$$

First we analyze the case $C(t_i) > 0$.

$$\bar{E}^{LSF}_{t_{i+1}} = \mathbb{E}\big(-C(t_{i+1})\big|C(t_{i+1}) < 0\big) = \frac{\mathbb{E}\big(-C(t_{i+1})\mathbf{1}_{\bar{B}_i}\big)}{\mathbb{P}(B_i)}$$

Considering each time step $(t_{i+1} - t_i)$ equal to $\frac{T}{n}$ we compute the expectation in the numerator as follows:

$$\mathbb{E}\big(-C(t_{i+1})\mathbf{1}_{\bar{B}_i}\big) = \mathbb{E}\Bigg(-\Bigg(C(t_i)\Bigg(m\frac{S(t_{i+1})}{S(t_i)} + (1-m)e^{r(t_{i+1}-t_i)}\Bigg) - (1-c)\zeta(t_{i+1})\Bigg)\mathbf{1}_{\bar{B}_i}\Bigg)$$

$$= -C(t_i)\Bigg(m\mathbb{E}\Bigg(\frac{S(t_{i+1})}{S(t_i)}\mathbf{1}_{\bar{B}_{i+1}}\Bigg) + (1-m)e^{r\frac{T}{n}}\mathbb{E}(\mathbf{1}_{\bar{B}_i})\Bigg) - \mathbb{E}\Big((1-c)\zeta(t_{i+1})\mathbf{1}_{\bar{B}_i}\Big)$$

From the proof of (3.90) it is clear that

$$\mathbb{E}(\mathbf{1}_{\bar{B}_i}) = \mathbb{P}(\bar{B}_i) = \bar{\mathbb{P}}^{LSF}_{t_i,t_{i+1}}$$

Using comparable calculations to the previous section of the proof, it is determined that

$$\mathbb{E}\Bigg(\frac{S(t_{i+1})}{S(t_i)}\mathbf{1}_{\bar{B}_i}\Bigg) = e^{\mu_S\frac{T}{n}}\Phi(\bar{d} - \sigma_S\sqrt{\frac{T}{n}}) \tag{3.94}$$

Now we computer the last expectation

$$\mathbb{E}\Big((1-c)\zeta(t_{i+1})\mathbf{1}_{\bar{B}_i}\Big) = \mathbb{E}\Big((1-c)\zeta(t_i)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I\left(W(t_{i+1}) - W(t_i)\right)}\mathbf{1}_{\bar{B}_i}\Big)$$

$$= (1-c)\zeta(t_i)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n}}\mathbb{E}\Big(e^{\sigma_I\left(W(t_{i+1}) - W(t_i)\right)}\mathbf{1}_{\bar{B}_i}\Big)$$

we follow the same logic used for the previous proof and we obtain

$$\mathbb{E}\Big(e^{\sigma_I\left(W(t_{i+1}) - W(t_i)\right)}\mathbf{1}_{\bar{B}_i}\Big) = e^{\frac{\sigma_I^2}{2}\frac{T}{n}}\Phi(\bar{d} - \sigma_S\sqrt{\frac{T}{n}}) \tag{3.95}$$

then

$$\mathbb{E}\Big((1-c)\zeta(t_{i+1})\mathbf{1}_{\bar{B}_i}\Big) = (1-c)\zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi(\bar{d} - \sigma_I\sqrt{\frac{T}{n}})$$

Hence, using the previous assumptions, and setting

$$F_2 = me^{\mu_S\frac{T}{n}}\Phi\Bigg(\bar{d}(t_i) - \sigma_S\sqrt{\frac{T}{n}}\Bigg) + (1-m)e^{r\frac{T}{n}}\bar{\mathbb{P}}^{LSF}_{t_i,t_{i+1}}$$

we have proved that the local expected shortfall for $C(t_i) > 0$ is equal to

$$\bar{\mathbb{E}}^{LSF}_{t_{i+1}} = \frac{-C(t_i)F_2 - (1-c)\zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi\Big(\bar{d}(t_i) - \sigma_I\sqrt{\frac{T}{n}}\Big)}{\bar{\mathbb{P}}^{LSF}_{t_i,t_{i+1}}}$$

Finally for the proof of $\mathbb{E}^{LSF}_{t_{i+1}} = \mathbb{E}\big(-C(t_{i+1})|C(t_{i+1}) < 0\big)$ in the case which $C(t_i) \leq 0$ we introduce another event

$$\bar{E}_i = \Big\{C(t_i)e^{r\frac{T}{n}} + (1-c)\zeta(t_{i+1}) < 0\Big\}$$

Hence

$$\bar{\mathbb{E}}^{LSF}_{t_{i+1}} = \frac{\mathbb{E}\Big(\big(-C(t_i)e^{r\frac{T}{n}} - (1-c)\zeta(t_{i+1})\big)\mathbf{1}_{E_i}\Big)}{\mathbb{P}(E_i)}$$
$$= \frac{-C(t_i)e^{r\frac{T}{n}} - \mathbb{E}\Big((1-c)\zeta(t_{i+1})\mathbf{1}_{E_i}\Big)}{\mathbb{P}(E_i)}$$

In this case, where $C(t_i) \leq 0$, the probability in the denominator is equal to:

$$\mathbb{P}(\bar{E}_i) = \mathbb{P}\Big(C(t_i)e^{r\frac{T}{n}} + (1-c)\zeta(t_{i+1}) < 0\Big)$$
$$= \mathbb{P}\Big(C(t_i)e^{r\frac{T}{n}} + (1-c)\zeta(t_i)e^{\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n} + \sigma_I\left(W(t_{i+1}) - W(t_i)\right)} < 0\Big)$$
$$= \mathbb{P}\Big(e^{\sigma_I\left(W(t_{i+1}) - W(t_i)\right)} < -\frac{C(t_i)}{(1-c)\zeta(t_i)}e^{r\frac{T}{n}}e^{-\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n}}\Big)$$

taking logarithm

$$= \mathbb{P}\Big(\sigma_I(W(t_{i+1}) - W(t_i)) < \ln\big(-\frac{C(t_i)}{(1-c)\zeta(t_i)}e^{r\frac{T}{n}}e^{-\left(\mu_I - \frac{\sigma_I^2}{2}\right)\frac{T}{n}}\big)\Big)$$
$$= \mathbb{P}\Big(\frac{(W(t_{i+1}) - W(t_i))}{\sqrt{\frac{T}{n}}} < \frac{1}{\sigma_I\sqrt{\frac{T}{n}}}\Big(\ln\Big(-\frac{C(t_i)}{(1-c)\zeta(t_i)}\Big) - (\mu_I - \frac{\sigma_I^2}{2} - r)\frac{T}{n}\Big)\Big)$$
$$= \Phi\left(\frac{\ln\Big(-\frac{C(t_i)}{(1-c)\zeta(t_i)}\Big) - (\mu_I - \frac{\sigma_I^2}{2} - r)\frac{T}{n}}{\sigma_I\sqrt{\frac{T}{n}}}\right) = \Phi(\bar{d}(t_i))$$

and in the same way that we compute $\mathbb{E}\Big((1-c)\zeta(t_{i+1})\mathbf{1}_{\bar{B}_i}\Big)$ it holds that

$$\mathbb{E}\Big((1-c)\zeta(t_{i+1})\mathbf{1}_{\bar{E}_i}\Big) = (1-c)\zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi(\bar{d}(t_i) - \sigma_I\sqrt{\frac{T}{n}})$$

Finally

$$\bar{\mathbb{E}}^{LSF}_{t_{i+1}} = -C(t_i)e^{r\frac{T}{n}} - \frac{(1-c)\zeta(t_i)e^{\mu_I\frac{T}{n}}\Phi\Big(\bar{d}(t_i) - \sigma_I\sqrt{\frac{T}{n}}\Big)}{\Phi(\bar{d}(t_i))}$$

□

3.4 Numerical application

We simulate the performance of the CPPI strategy shown in the previous chapter using Monte Carlo simulations to analyze its behavior. Clearly, for the simulations, a discretization of the interval $[0;T]$ is required, thus we will only refer to the

conclusions gained in the preceding chapter regarding the discrete time treatment. The interval under consideration will be divided into $n = T * 12$ intervals, resulting in a monthly discretization. Assume that contributions are paid at monthly intervals, and that the instant of payment corresponds to the instant of rebalancing. These instants coincide to the n points along the period $[0;T]$.
To proceed with simulations of the fund value we use the Euler scheme.

The Euler scheme Consider a generic diffusion process:

$$dY_t = a(Y_t, t)dt + b(Y_t, t)dW_t \qquad\qquad Y_{t_0} = Y_0$$

with unknown solution. The Euler scheme allows us to compute numerically the distribution of $Y_t,\ for\ t \geq t_0$.
The scheme involves dividing the interval $[0;T]$ into n sub-intervals defined by $n+1$ times $\{0, t_1; t_2, ..., t_n = T\}$, and then using the estimator $\widehat{Y}_t$ to approximate the "exact" process Y_t, using the relation:

$$\widehat{Y}_{t_{i+1}} = \widehat{Y}_{t_i} + a(\widehat{Y}_{t_i}, t_i)(t_{i+1} - t_i) + b(\widehat{Y}_{t_i}, t_i)\sqrt{(t_{i+1} - t_i)}\varepsilon_{i+1}$$

where $\varepsilon_{i+1} \sim N(0,1)$, with $i = 0, 1, 2, ..., (n-1)$.
In the course of the thesis we considered the time step $t_{i+1} - t_i$ constant, i.e.

$$t_{i+1} - t_i = \frac{T}{n}, \qquad \forall\ i = 0, 1, 2, \ldots, n$$

therefore the above equation simplifies to:

$$\widehat{Y}_{t_{i+1}} = \widehat{Y}_{t_i} + a(\widehat{Y}_{t_i}, t_i)\frac{T}{n} + b(\widehat{Y}_{t_i}, t_i)\sqrt{\frac{T}{n}}\varepsilon_{i+1}$$

We take into account the same fund dynamics (3.50)-(3.51) both for NPV floor and for random floor[5].

The proposed thesis explores an asset and liabilities management methodology suitable for pension funds. Asset Liability Management, as defined in the first chapter (section 1.4.1), is a set of procedures and processes that support management decisions in the integrated management of pension fund assets and liabilities. The goal is to maximize the trade-off between expected return and assumed risk based on the available knowledge and future scenarios. Asset and liability management can include a variety of operations, including strategic asset allocation.
Asset allocation is the technique of allocating capital among various investment classes in order to build a diversified portfolio that is as responsive to the needs of the investor (pension fund member) in terms of risk-return trade-offs as possible.
Asset allocation aims at identifying portfolios that are optimal for covering social security commitments, consistent with the riskiness target of the sector. The reference time horizon is $[0;T]$. The asset classes taken into account in the application are the following:

[5] we have already seen that in both cases we have the same fund dynamics

ISIN	Fund
IT0001223095	Anima Obbligazionario Corporate A
IT0001040093	Anima America A
IT0001015921	Anima Pianeta A
IT0000386026	Anima Valore Globale A
IT0000380722	Anima Sforzesco A
IT0000380706	Anima Visconteo A

Figure 3.1. The table shows the ISIN code and the name of the ANIMA fund to which we will refer for the numerical application.

It proceeds with an estimate of the volatility on each asset class and an estimate of the expected returns on each asset class; the estimates are based on the historical series available (the data from 2018 to 2020 have been used). From these elements we define the set of efficient strategies (efficient frontier) that we can see in the image below:

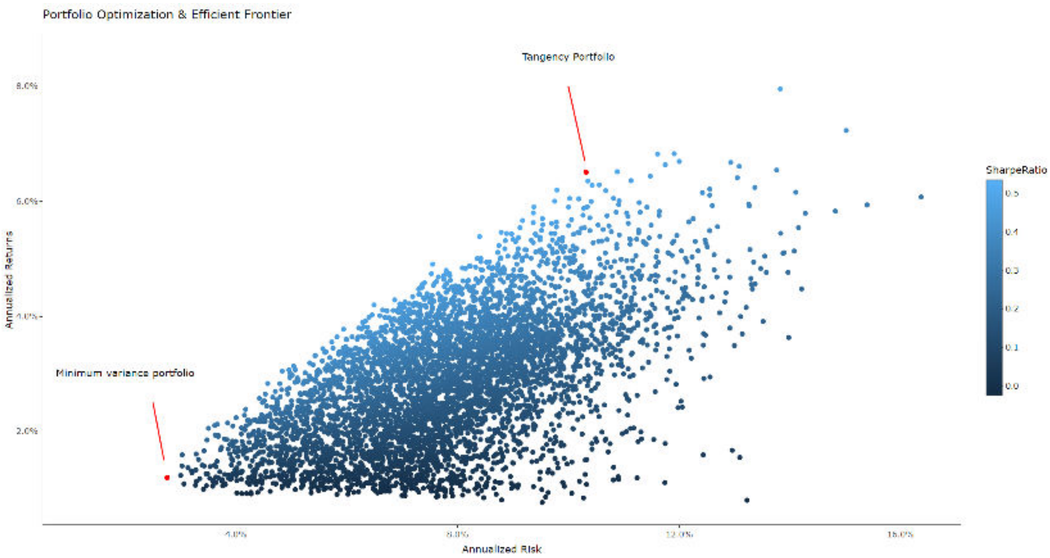

Figure 3.2. Each point on the graph represents a combination of the securities listed above. Each combination represents a risk/reward pair. Portfolio with a low variance and tangency portfolio are illustrated in red. The darker spots will have lower Sharpe ratio, the lighter ones higher Sharpe ratio. The Sharpe ratio is a measure that relates the difference between the portfolio's return and the return on a risk-free investment (which we assume is equal to 0.01) with the portfolio's riskiness.

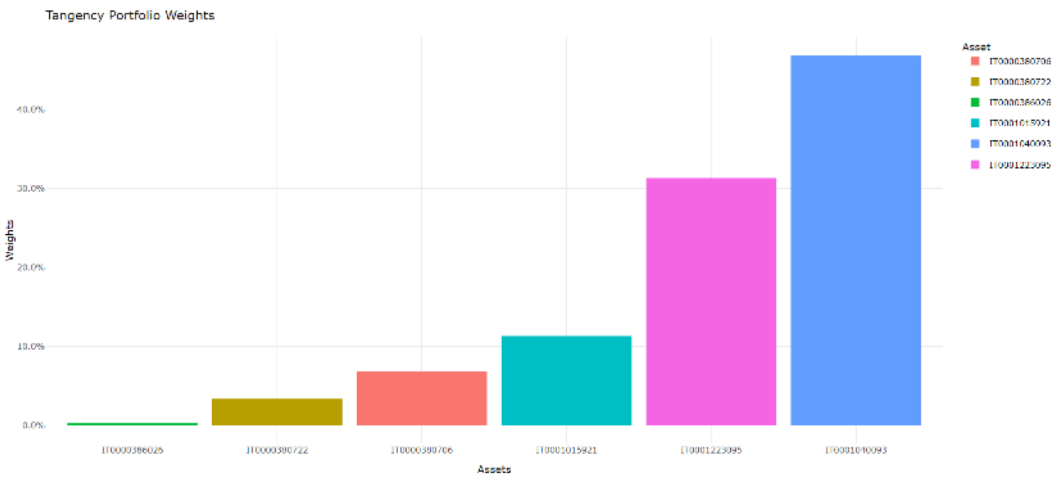

Figure 3.3. Percentage of capital that will be invested in each fund to build the tangency portfolio.

Return	Risk	Sharpe Ratio
0.0651	0.1032	0.4503

Figure 3.4. The table shows return, risk and Sharpe ratio of the tangency portfolio

We use the asset allocation of the tangency portfolio, which is identified by the efficient frontier's point of tangency. The weights assigned to each fund in this portfolio are shown in 3.3 and the corresponding Return, Risk and Sharpe Ratio values are shown in 3.4.

With this information we can calibrate the risky component of our portfolio. The following table lists the input parameters. Unless otherwise mentioned, the given parameter set will be used in numerical computations throughout this section.

Parameter	Symbol	Value
Stock drift	μ_S	0.0651
Stock volatility	σ_S	0.1032
Stock initial value	S_0	43.24
Labor income drift	μ_I	0.006
Labor income volatility	σ_I	0.07
Labor income initial value	I_0	40
Riskless asset	R_0	10
Maturity	T	10
Multiplier	m	6
Guarantee rate	c	0.7
Contribution rate	ζ	0.1

For the calculation of the sport rate r we will use the structure provided by EIOPA for the basic risk free rate curves as of 11/30/2021[6]. For more information on the construction of the risk free rate curve by EIOPA see Appendix C.

The contributions are received on a monthly basis, therefore we partition the range [0;T] so that the calibrations occur on a monthly basis and correspond with the payment of contributions.

Given the parameters, we run simulations of the portfolio's possible trend. As previously stated, we will apply the Eulero scheme established in the preceding paragraph for the dynamics of the value of Y_t. The portfolio is partially invested in the tangency portfolio found by asset allocation, with the remainder invested in a non-risky component that evolves at a constant rate r. The tangency portfolio (the risky component) moves like a geometric Brownian motion (as defined for S_t) for which the solution in closed form is known, and the same is true for the labor income process. We can create portfolio dynamics using these components. We show an example of a potential trajectory for both the NPV and random floors.

It is interesting to note that in the scenario of NPV floor, the fund begins with a lower value of the bond floor. This is because, the bond floor begins with a percentage of all contributions discounted at starting time 0, the fund begins with the value of the first contribution (equal to ζI_0), which is naturally smaller than the prior value. The fund, on the other hand, continues to feed over time with monthly contributions provided by members, resulting in an increase that allows Y_t to exceed B_t in a given $t > 0$. As long as the fund does not exceed the bond floor, the exposure is zero, which means that the whole fund, including new contributions, will be invested in the non-risky component. This reduces the potential rewards from investing in the riskier component. The bond floor, shown in red on the graph, is decreasing as a result of EIOPA's estimated negative rates. In the scenario of a random floor, the fund begins at a greater value than the bond floor. Both values (Y_t and B_t) increase stochastically and are increased by monthly contributions. The

[6]Available at https://www.eiopa.europa.eu/tools-and-data/risk-free-interest-rate-term-structures_en

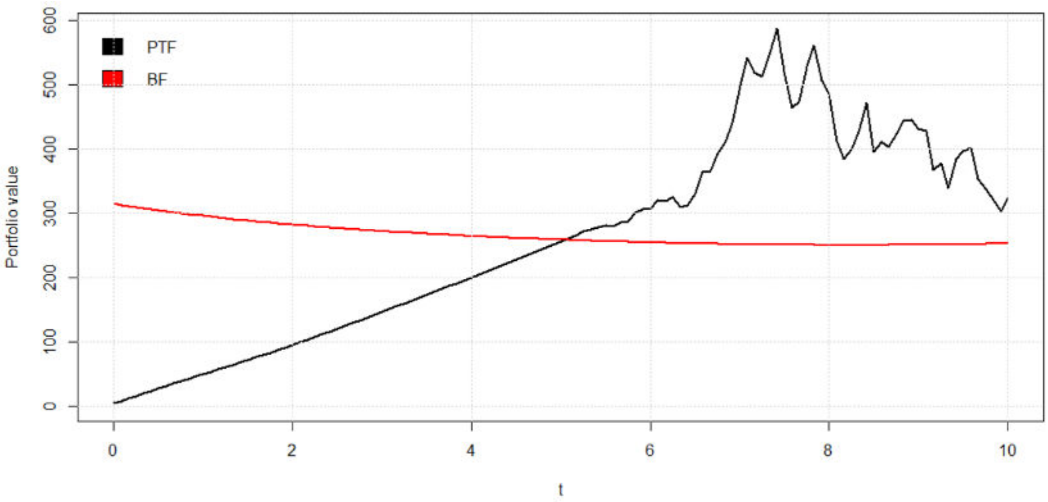

Figure 3.5. An example of a trajectory for the CPPI strategy with an NPV floor

bond floor, on the other hand, rises at a deterministic rate, and the fund is also invested in a risky component that provides larger earning opportunities.

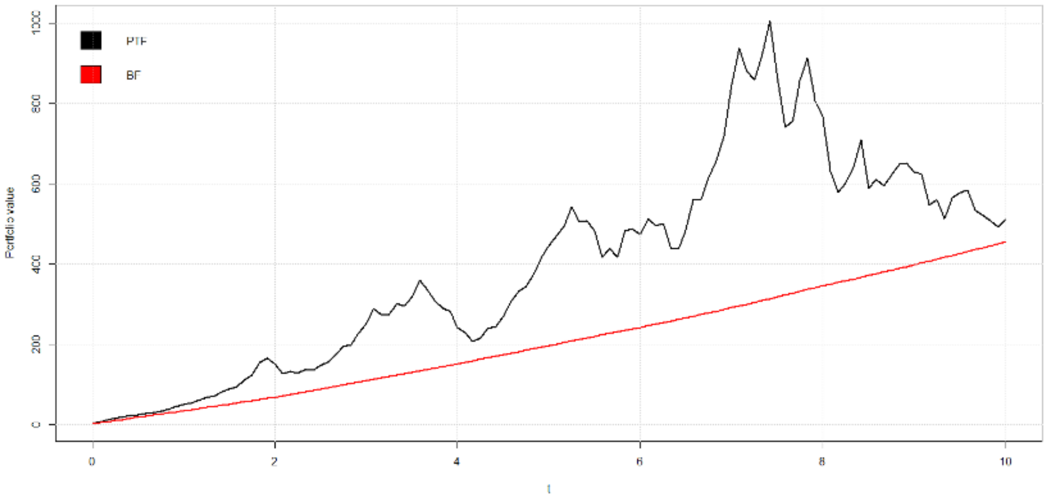

Figure 3.6. An example of a trajectory for the CPPI strategy with a random floor

We simulate 10.000 trajectories in order to thoroughly examine and compare the behavior of the strategies presented. We compare the two strategies presented in the thesis, beginning with the simulated distribution of the fund's final value for the NPV floor and the random floor. So let's compare the histograms of the final values, Y_T, for the two floors:

We can clearly see that in the case of NPV floor, we have larger frequencies for lower fund value at maturity, as well as a shorter right tail, indicating that there is no evidence of extremely high earning potential.

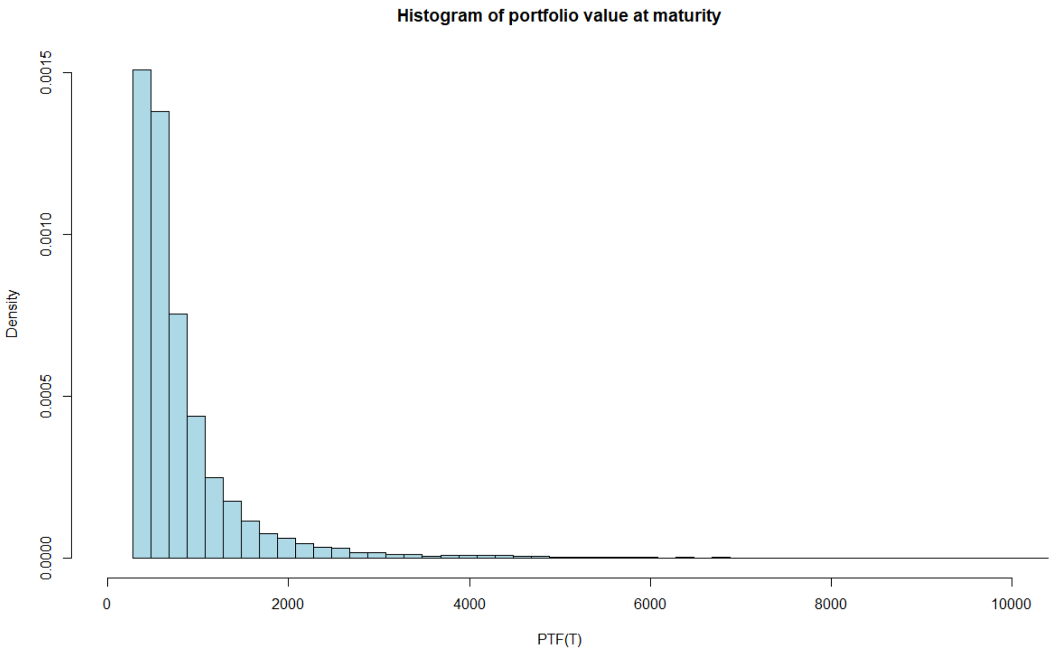

Figure 3.7. Distribution of the fund's final value with NPV floor based on 10.000 simulations

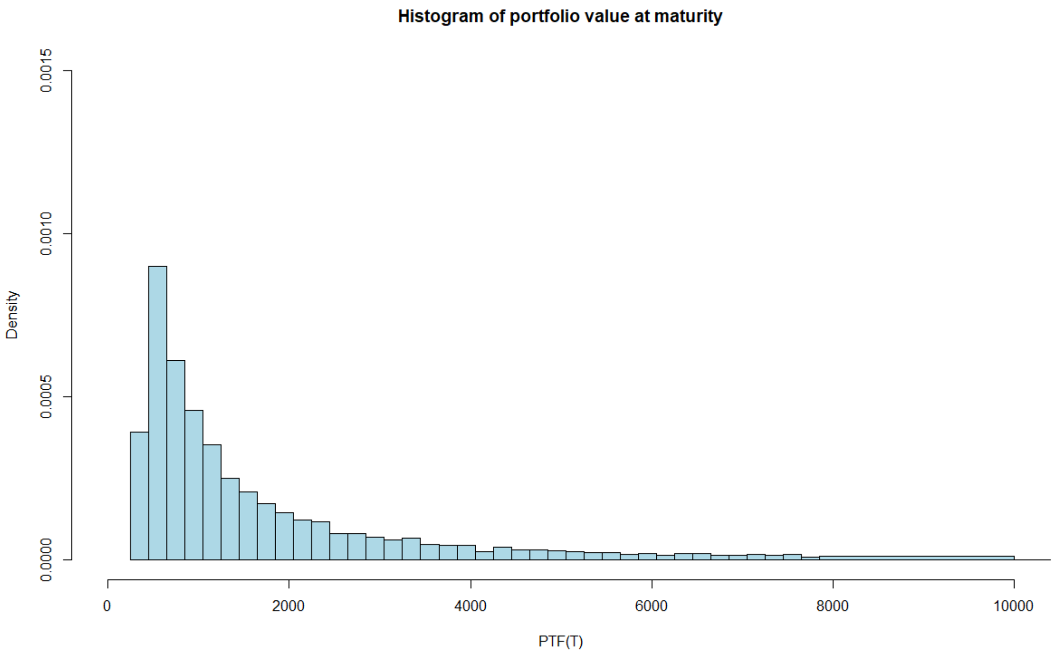

Figure 3.8. Distribution fund's final value with random floor based on 10.000 simulations

In the context of the CPPI strategy with random floor, we can notice a considerably longer right tail than in the prior scenario. This suggests higher average profits, as well as lower frequencies for values where the strategy concentrates on the NPV floor and higher frequencies for later values.

Let's look at some synthesis measures of the fund's value in T for the NPV floor and the random floor, such as the expected value (EV), standard deviation (SD), coefficient of variation (calculated as the ratio of standard deviation to expected value) (CV), and the minimum (Min) and maximum (Max) values assumed in our simulation.

Floor	EV	SD	CV	Min	Max
NPV	832.53	750.66	0.9	279.35	18946.51
Random	2192.09	6895.392	3.15	243.90	383684.7

As shown in Figures 3.7 and 3.8, the CPPI approach with random floor appears to have a greater chance of high gains, and it has a much bigger variability than the NPV floor strategy.

However, in order to better understand the trend and effectiveness of the two approaches, let us examine local behavior using the risk measures outlined in Section 3.3.4 of this chapter. We will also analyze the behavior when the multiplier changes in order to understand how sensitive this technique is to an increase in m, which, as previously stated, implies more risk. As a result, tracing the evolution of calculated measurements is the best way to understand them.

Let's first analyze the trend of the local cash-lock probability. We calculate the local probability of cash lock from the 10.000 simulated trajectories at all moment: for all t_i, we calculate the probability of cash lock 10.000 times and average for every moment, so we obtain the trend of the aforementioned probability. We repeat the same logic with different values of m to see how it changes as the multiplier increases. We proceed in the same manner with the other risk measures.
Let us now look at the graphs associated with this trend for the two strategies described: the NPV and the random floor

We can see that regular inflows provide good protection against cash-lock, as illustrated in Figure 3.9 and 3.10. The maximum probability is bounded above even for a very high multiplier value of 15.
In particular for the case of NPV floor, it is evident that the probability remains equal to one for the entire period in which the portfolio is below the floor and, as a result, it is only invested in risk-free business. When the value of Y_t exceeds the bond floor, the cash lock probability decreases until it approaches 0 for lesser multipliers. For higher values of m, we notice that the chance of cash lock grows in the last period, both for the NPV floor and for the random floor. Considering $m = 15$, the maximum value reached by the probability in the first strategy is equal to 0.0484 and in the second case is equal to 0.198. The CPPI approach with an NPV floor has lower cash lock probabilities towards maturity but higher for the remainder of the period considered, whereas the random floor strategy maintains a low amount of risk; particularly for multiplier levels less than 10, the probability of cash lock is negligible.

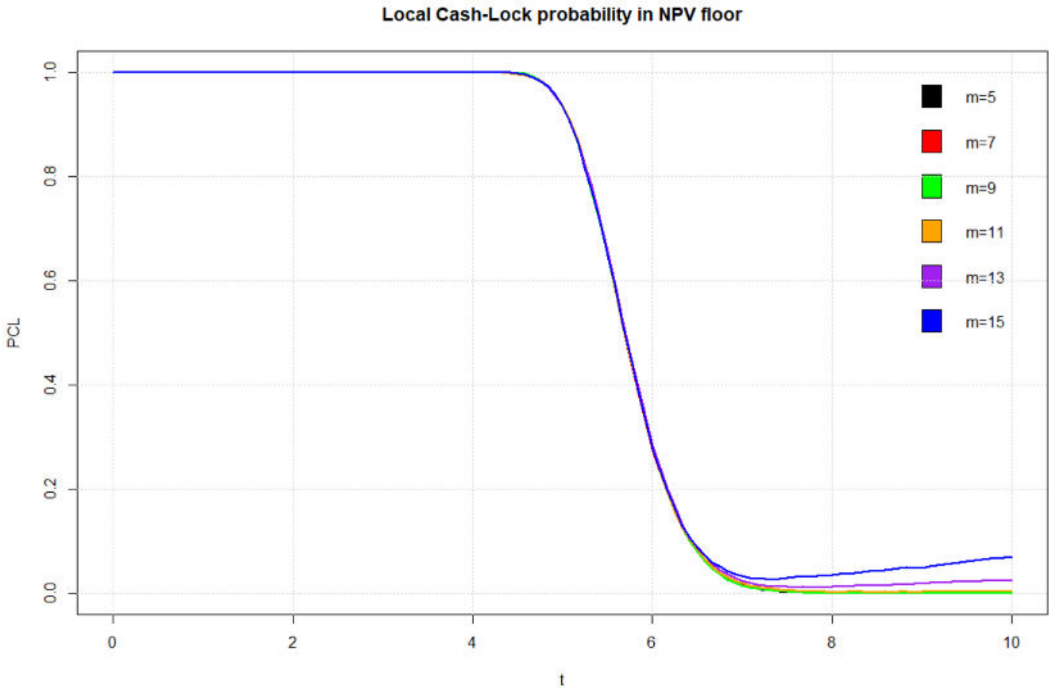

Figure 3.9. Cash-lock probability trajectories in a CPPI strategy with NPV floor during the period considered $[0; 10]$.

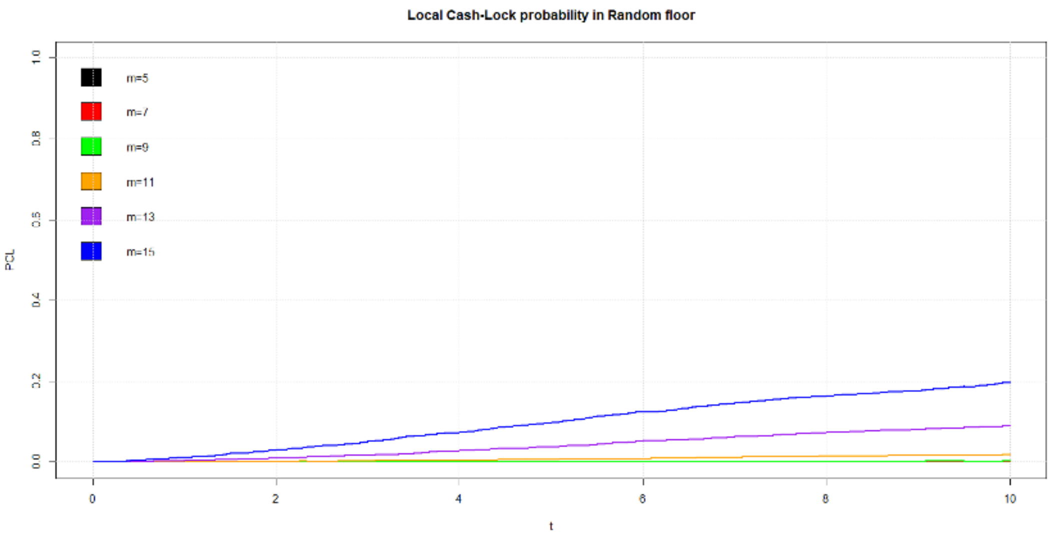

Figure 3.10. Cash-lock probability trajectories in a CPPI strategy with random floor during the period considered $[0; 10]$.

The local shortfall probability follows the same logic as the cash lock probability just analyzed. In reality, even in this case, for the CPPI strategy with NPV floor, the probability that the fund is below the bond floor is certain until the fund's value reaches the bond floor. As long as the portfolio is below the bond floor, the value of m has no effect on the trend of this probability since all capital is invested in the non-risky sector. After $Y_t = B_t$, the probability decreases towards 0. During this period until maturity, the probability is sensitive to changes in m, especially for values larger than 11 as maturity approaches. Considering at the approach with random floor, we see that the behavior is analogous to the prior measure examined. As maturity approaches, there is an increasing probability of a shortfall for m values larger than 11, but for lesser values, the probability is negligible.

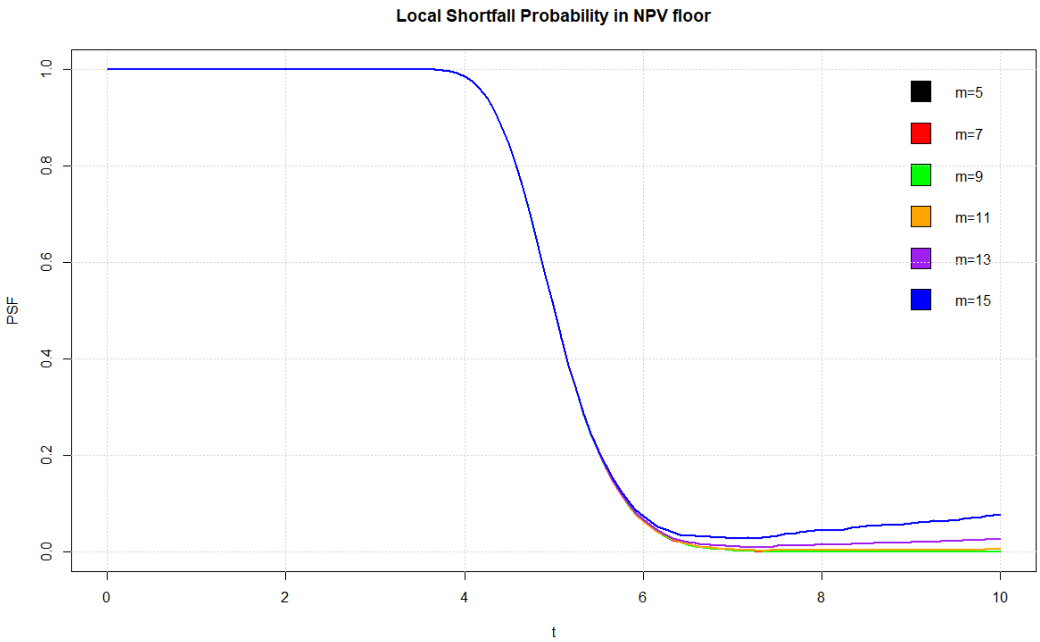

Figure 3.11. Shortfall probability trajectories in a CPPI strategy with NPV floor during the period considered [0; 10].

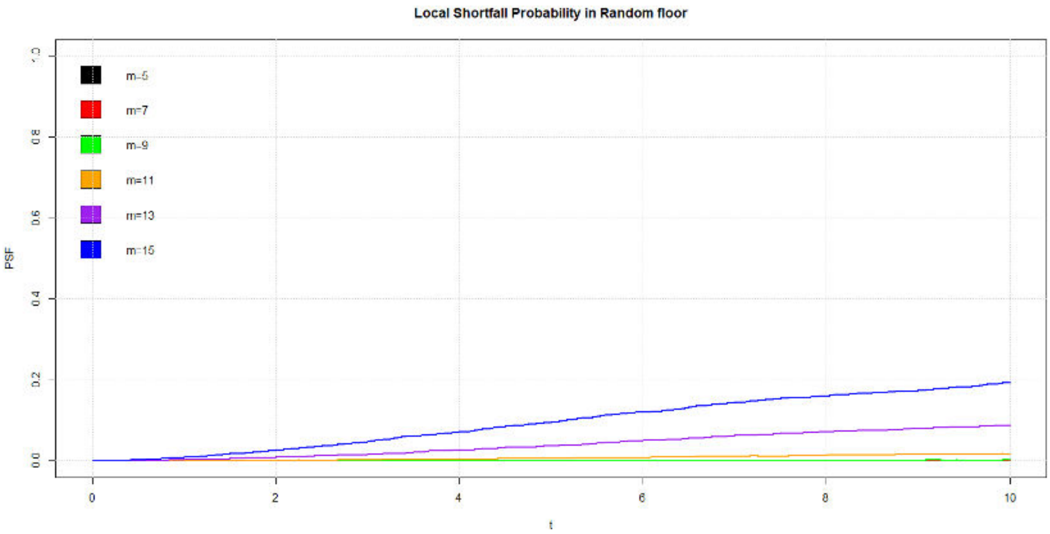

Figure 3.12. Shortfall probability trajectories in a CPPI strategy with random floor during the period considered [0; 10].

After calculating the risk based on the distance between the portfolio value and the minimum value per trajectory, we calculate the value of the prospective loss that the manager may experience if the portfolio value falls below the bond floor for each moment studied. Therefore let's see the graphs related to local expected shortfall for both floor.

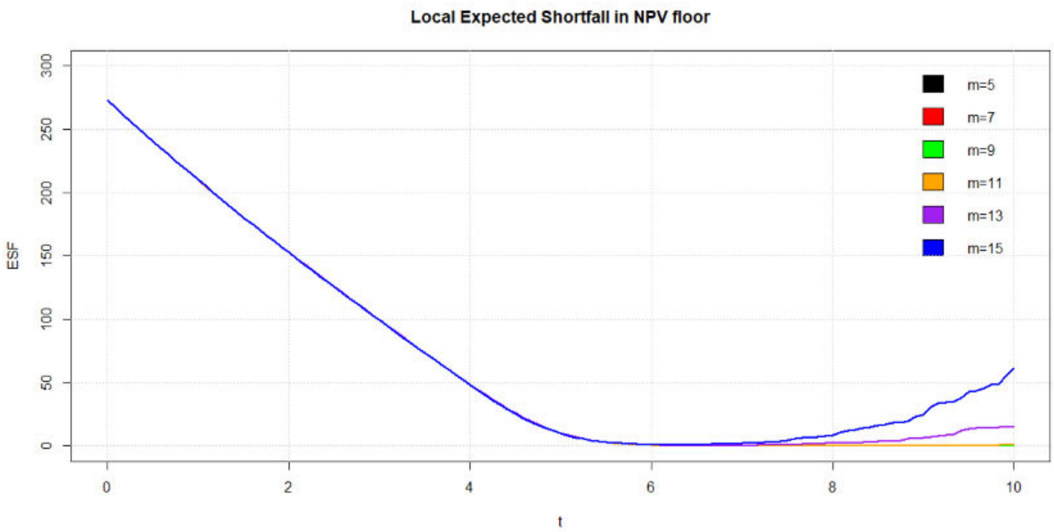

Figure 3.13. Expected shortfall trajectories in a CPPI strategy with NPV floor during the period considered [0; 10].

The predicted loss in the case of an NPV floor is higher at the start of the period, but this is mitigated by the continual payment of contributions, which raises the fund's value above the bond floor. For low m values (less than 10), the projected shortfall has a decreasing tendency towards zero until the end of the period; for

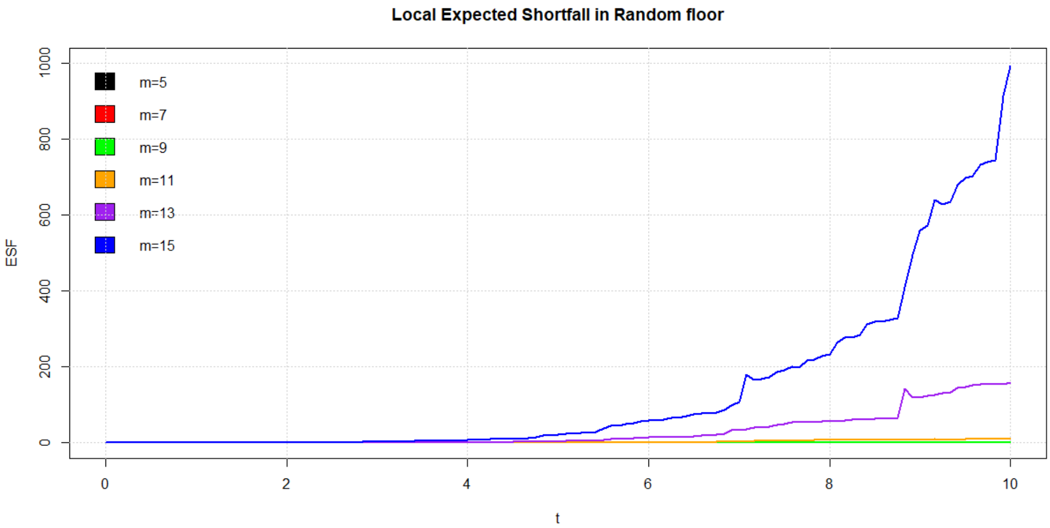

Figure 3.14. Expected shortfall trajectories in a CPPI strategy with random floor during the period considered [0; 10].

larger m values, the reduction is slower, and there is a new moderate increase in the last year. In the approach with random floor, the expected shortfall remains zero for m values less than 10. For m values more than 11, the expected loss is greater and develops with time until it reaches maximum values at maturity. In particular, the increase in the expected loss at maturity of $m = 15$, which reaches much higher levels than the other cases considered, is significant.

Finally, as the multiplier changes, we examine the final values for the average (expected value) and variability (coefficient of variation) of the fund value, as well as the corresponding gap-risk measures, i.e. the probability of shortfall and the expected shortfall.

NPV floor				
Multiplier	Shortfall Probability	Expected Shortfall	Expected Value	CV
5	0	0	811.01	0.853
7	0	0	1209.24	1.68
9	7e-04	0.101	1788.13	3.198
11	0.0047	1.0023	2738.13	5.77
13	0.034	16.493	4258.145	9.78
15	0.111	77.501	6604.35	15.898

Random floor				
Multiplier	Shortfall Probability	Expected Shortfall	Expected Value	CV
5	0	0	1239.74	1.51
7	0	0	2330.45	4.99
9	0.0017	0.1606	5119.84	13.102
11	0.0172	9.9403	12231	24.57
13	0.0884	158.32	27710.71	37.48
15	0.1942	994.782	52252.97	58.73

We see that, for each multiplier value, the shortfall probability and the expected shortfall at maturity is always smaller in the case of CPPI with an NPV floor than in the case of CPPI with a random floor. The probability of shortfall differs only to the second decimal digit in the two situations, but the difference in expected shortfall for the two processes is significantly larger for multiplier values greater than 10. Lower risk measurement values correspond to lower returns and variability. In particular, for both floor processes, we can observe a significant rise of the expected value and the coefficient of variation at maturity to the increase of the value of m. This rise is particularly noticeable in the case of random floor, where the average value almost doubles for each multiplier value. You obtain very huge values for final wealth but also very large values for potential loss. For m>10, we have greater expected shortfall values at maturity, as well as higher expected value and coefficient of variation values, but only by considerably lesser amounts and those created by the random floor.

We now wish to compare the results acquired with those that would have been produced if, instead of utilizing the tangency portfolio, we had chosen the portfolio with the minimum variance (shown in the figure 3.2).
We now illustrate the weights applied to each asset class in order to form this portfolio (in Figure 3.15), as well as the metrics associated with it and how the simulation settings change.

Return	Risk	Sharpe Ratio
0.0109	0.0294	0.064

The reduction in portfolio's riskiness certainly implies a decrease in the performance measurements. Consider the parameters presented at beginning of this section, with the exception of the parameters related to the risky component, for which the following values are used:

Parameter	Symbol	Value
Stock drift	μ_S	0.0109
Stock volatility	σ_S	0.0294
Stock initial value	S_0	31.01

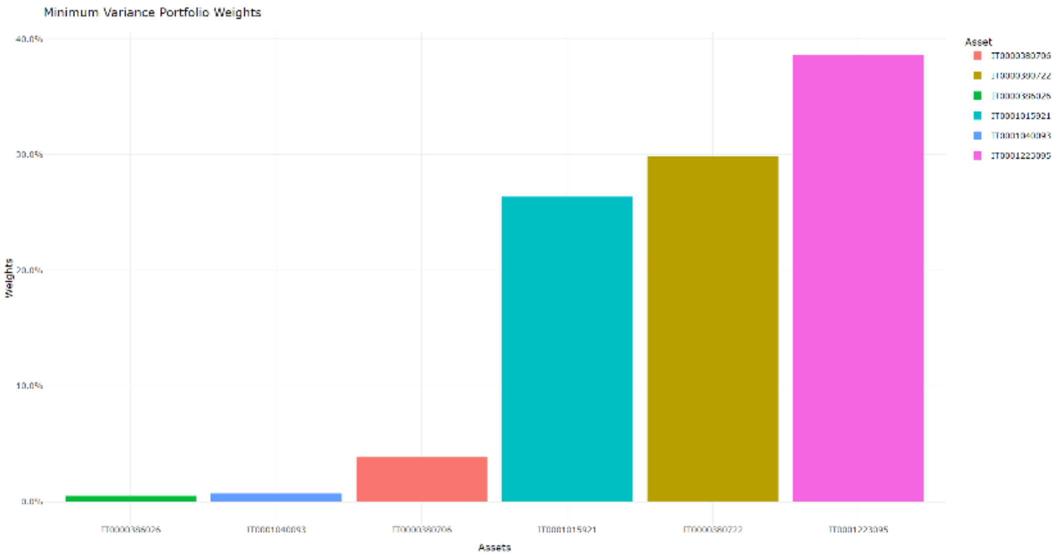

Figure 3.15. Percentage of capital that will be invested in each fund to build the tangency portfolio.

As an illustration, in this scenario, we display a trajectory for each floor investigated process, taking into account the investment in a minimum variance portfolio. Already from the first observation, we can see a more linear course for the value of the fund. The peaks upwards or downwards are smoother, and in the case of the strategy with NPV floor, we can see that the fund value reaches the bond floor earlier than we saw in Figure 3.5. Such assertions cannot, of course, be based solely on the observation of a single trajectory. Many trajectories have been simulated and compared; however, for the purposes of this explication, we will just use the following as an example.

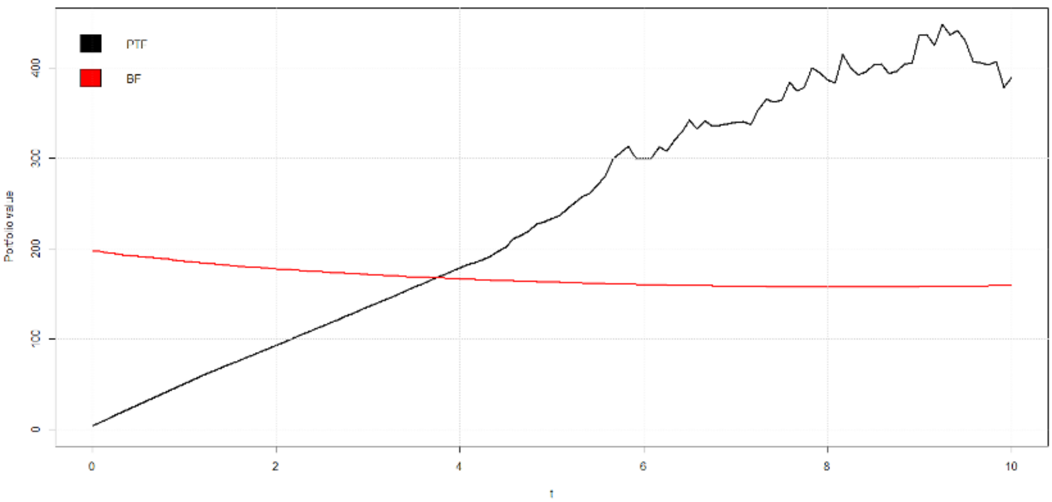

Figure 3.16. An example of a trajectory for the CPPI strategy with an NPV floor

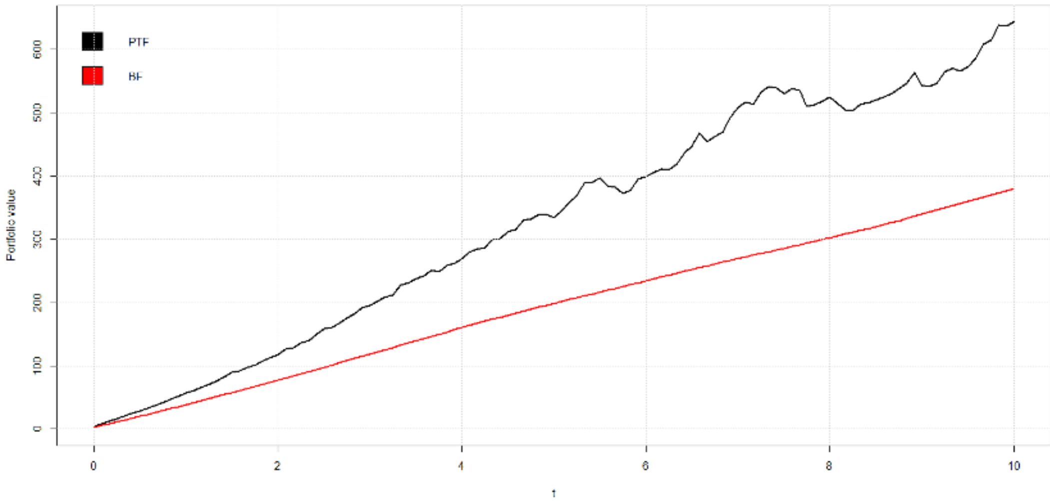

Figure 3.17. An example of a trajectory for the CPPI strategy with an NPV floor

Again we simulate 10.000 trajectories. As a result, we examine the distribution of fund's value at maturity $T = 10$ as well as the associated measurements (Expected value, Standard deviation, coefficient of variation, minimum and maximum value).

Floor	EV	SD	CV	Min	Max
NPV	574.11	147.56	0.26	260.79	1624.86
Random	612.35	140.33	0.23	288.25	1571.32

In this scenario the adoption of different processes for the bond floor has a negligible influence. The histograms in Figures 3.18 and 3.19 show that the two distributions are not significantly different. Even when the synthesis values are compared, we can notice no significant differences; the method with random floor now has a marginally higher average but also a slightly lower variability. When we compare these outcomes to those produced using the tangency portfolio, it is clear a significant decrease in return. The values assumed by the fund at maturity using the tangency portfolio's asset allocation are considerably higher; this is especially evident for the method with random floor, whose returns are three times more than those assumed using the portfolio with the minimum variance. As expected, this decline is accompanied by a sharp decrease in the variability of the portfolio value distribution. In particular, for the strategy with NPV floor, considering the portfolio with the minimum variance, the coefficient of variation is less than one-third of the same measure obtained by using the tangency portfolio. This discrepancy is amplified in the case of random floor, where the coefficient of variation is 13 times smaller than the CV calculated previously.

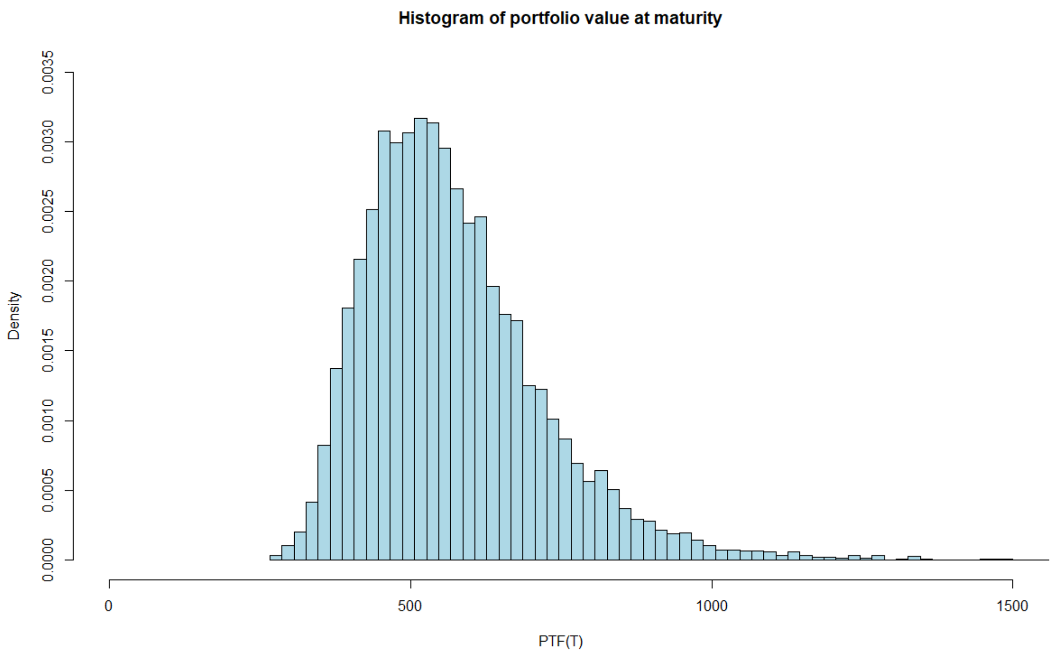

Figure 3.18. Distribution of the fund's final value with NPV floor based on 10.000 simulations

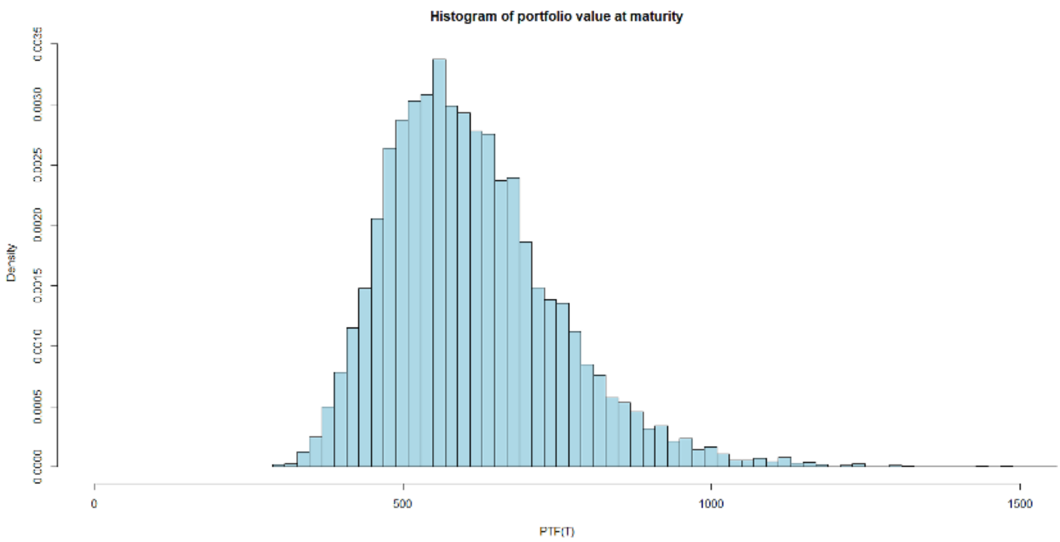

Figure 3.19. Distribution of the fund's final value with random floor based on 10.000 simulations

Let's take a look at our strategy's local behavior, as we did with the tangency portfolio. So, to vary the multiplier, let's examine at the trajectory of the risk measurements discussed in the thesis.

About the local cash-lock probability for both floor processes

It is worth noting that increasing the multiplier has no influence on the trend

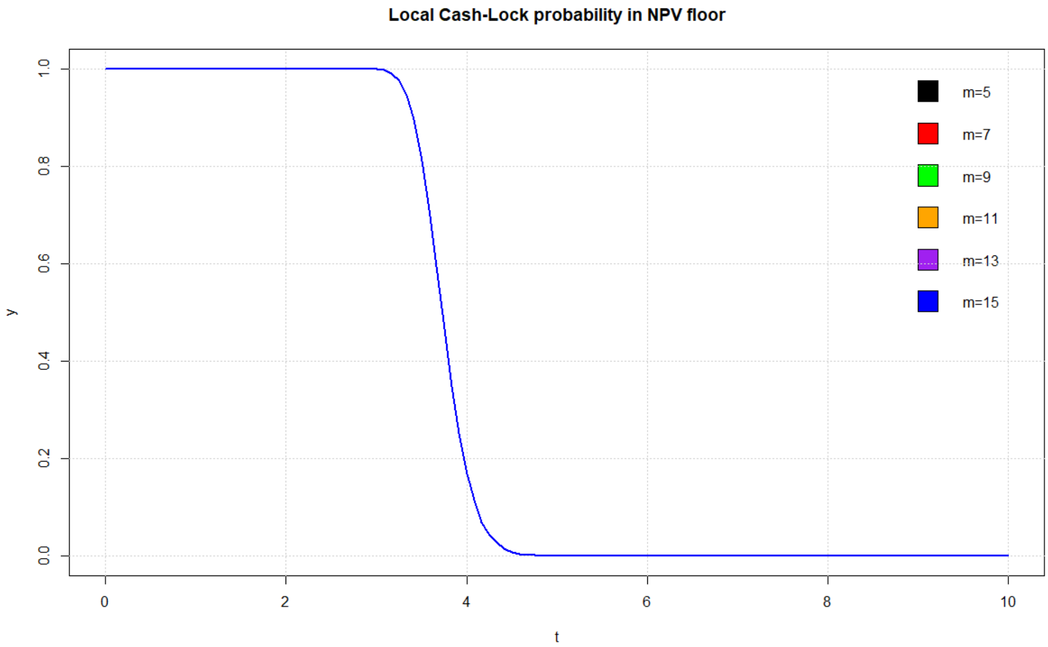

Figure 3.20. Cash-lock probability trajectories in a CPPI strategy with NPV floor during the period considered [0; 10].

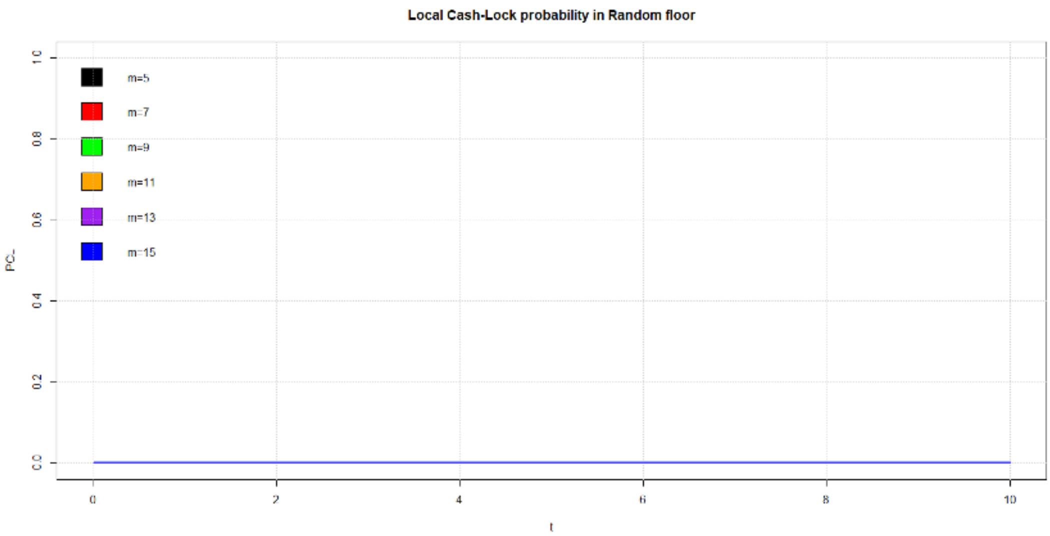

Figure 3.21. Cash-lock probability trajectories in a CPPI strategy with random floor during the period considered [0; 10].

of the probability of cash-lock in this scenario. Because the paths are completely equal, they overlap. This occurs for both CPPI with an NPV floor and CPPI with a random floor. Comparing graph 3.20 with Figure 3.9, we can see that the period in which the fund remains below the bond floor is less than in the previous case. For each multiplier value after the decrease, the cash lock probability remains zero until maturity. When investing in the tangency portfolio in the preceding scenario, there was an increase in probability towards the conclusion of the period for both the NPV floor and the random floor. The CPPI strategy with random floor now has zero cash lock probabilities for the whole period and for each multiplier. Because the minimum variance portfolio chosen for the risky segment is low risk, the strategy is resistant to changes in the multiplier. Some calculations show that only for multiplier values more than 35, the probability increases significantly.

We now present the graphs relating to the trend of the shortfall probability and the expected shortfall in the case of NPV floor. Again, varying m between 5 and 15 results in no change in the trajectory of risk measures. The shortfall probability begins to decline before the cash-lock probability, as previously discussed. The decrease occurs earlier, as expected, than the probability of shortfall in the case of tangency portfolio investment.Once it reaches 0, the probability remains zero until the end. The same analyses can be performed for the expected shortfall. It starts at a lower value than in the prior instance. As a result, the initial gap between the bond floor and the fund is less than the difference shown in Figure 3.13, and the fund takes "less time" to reach and overcome the bond floor. For the random floor, these measures are always 0 throughout the period and for each multiplier examined, as in figure 3.21.

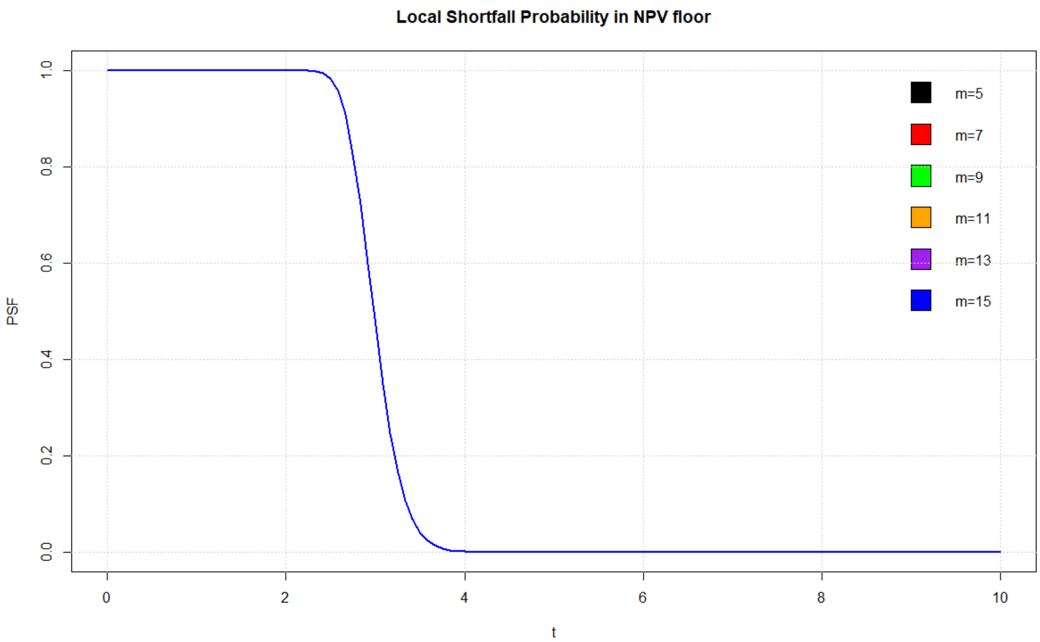

Figure 3.22. Shortfall probability trajectories in a CPPI strategy with NPV floor during the period considered $[0; 10]$.

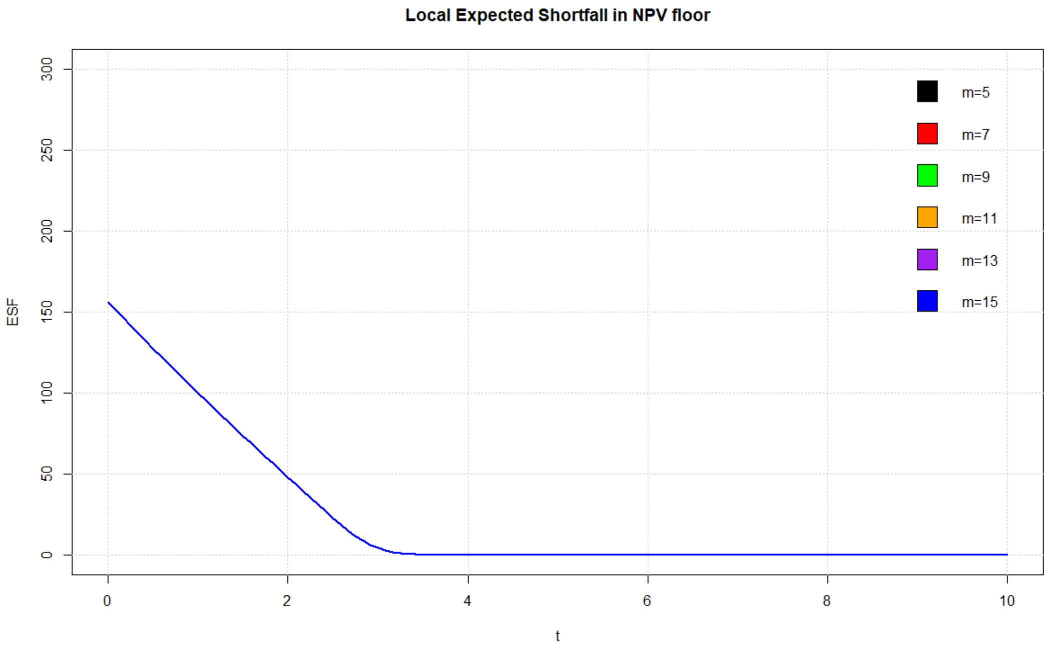

Figure 3.23. Expected shortfall trajectories in a CPPI strategy with NPV floor during the period considered $[0; 10]$.

Let us examine the behavior at maturity as the multiplier grows, as we did previously. The gap-risk measurements, as seen in the graphs, are all equal to 0 for each m values and for both floor processes. As expected, as the multiplier grows, so do the expected values and the coefficient of variance. It is worth noting that for m values between 7 and 9, the random floor performs better in terms of both performance and variability. The average final wealth values are higher for the other multiplier values (except for m=5), and the coefficient of variance is slightly larger. We remind you that the CPPI technique with random floor has zero risk measures throughout the time, so if we consider a fund's participant who is unwilling to wait for the fund's value to exceed the bond floor, the CPPI strategy with random floor always appears to be the best choice in this case.

		NPV floor		
Multiplier	Shortfall Probability	Expected Shortfall	Expected Value	CV
5	0	0	582.25	0.27
7	0	0	640.74	0.36
9	0	0	711.68	0.47
11	0	0	798.07	0.61
13	0	0	903.7	0.78
15	0	0	1033.34	0.99

Random floor				
Multiplier	Shortfall Probability	Expected Shortfall	Expected Value	CV
5	0	0	576.05	0.2
7	0	0	657.85	0.27
9	0	0	787.56	0.42
11	0	0	996.83	0.66
13	0	0	1339.32	1.02
15	0	0	1906.56	1.52

In conclusion, we have observed the strategy's behavior both when using a riskier portfolio (tangency portfolio) and when using a portfolio with lower variance (minimum variance portfolio).
The fund's value at maturity has been calculated in both scenarios and for each floor process specifically for a multiplier of 6 and, then, for m varying between 5 and 15. The local behavior of the risk measures has been investigated, as well as the effect of increasing the multiplier for each scenario. Let us now report on our conclusions and potential future advancements in the subject under examination.

Chapter 4

Conclusion

We examined the origins and evolution of supplementary pension plans in the first section of the paper, as well as the reasons why pension schemes are increasingly becoming an essential component of statutory pension systems in many countries. This thesis introduces various Constant Proportion Portfolio Insurance techniques with varied structures in defined contribution pension plans in which each investor pays consecutive stochastic payments based on its stochastic income. We present two bond floor processes, one that is directly dependent on stochastic contributions (the random floor) and the other on the time zero value of future payments (the Net Present Value floor). In particular, the various processes correspond to two CPPI guarantee concepts proposed on the basis of both continuous and discrete trading assumptions. As a result, the problem is addressed in both complete and incomplete markets. In a continuous trading situation, perhaps in a complete market, a portfolio that replicates the flow of future contributor payments is identified. In this case, the proposed strategy poses no risk; continuous recalibrating allows you to react instantly to market changes and, as a result, avoid any undesirable scenario. We have explained why these strategies cannot be applied realistically. As a result, we proceed to theoretically and analytically illustrate the scenario of discrete-time trading, and the we proceeded on to the derivation of risk measures.
Let us move on to the numerical applications to see how the proposed techniques are working and to evaluate their performance in various scenarios. We first proceed with the asset allocation of the asset class in figure 3.1, we select a portfolio with high returns but also high riskiness (tangency portfolio), as contrast to a portfolio with minimal variance and thus lower returns (minimum variance portfolio). We investigate the behavior of the CPPI approach with NPV floor and with random floor using Monte-Carlo simulations and Euler's technique for dynamics discretization. We fix the multiplier value and investigate the terminal asset distributions and performance of both strategies, first taking into account the risky investment in the portfolio tangency and subsequently in the minimum variance portfolio. Each strategy's performance will be compared based on the evolution of the risk measures and the values of the final wealth synthesis measures. The sensitivity of strategies to changes in the multiplier is also examined, with values of m ranging from 5 to 15. Let's start with the results for the situation where the risky sector is the tangency portfolio. By using these numerical results, it is possible to deduce that for smaller

m values, the CPPI with random floor is always the successful approach. It has superior profit margins and negligible possibilities of cash-lock and shortfall as long as the multiplier is less than 10. Exceeding the duration of time required for the fund's value to surpass the bond floor, even for the CPPI with NPV floor, the phenomena of cash-lock or shortfall is extremely rare (for not high multipliers). The NPV floor begins at a greater value than the fund, implying that the fund will only invest in non-risky sectors, resulting in lower returns and lesser variability. When we consider larger multiplier values, the approach that uses the random floor does not perform well, because the probability of cash-lock, the probability of shortfall, and, in particular, the expected shortfall are all extremely high in the last period. The CPPI method with NPV has a better hold, as the increases in risk measures are lower than in the other approach (always considering the past period when the fund is lower than the bond floor). As a result, for large m values, the CPPI approach with NPV floor is a superior alternative, with an intermediate tail and moderately high profits for participants who intend to remaining in the pension plan long enough to restore the portfolio from its initial negative position. Rather, with a medium-low multiplier, the CPPI with random floor outperforms in terms of trade-off returns and risk measures, although having a larger variability. Let us now examine what changes if the risky sector analyzed is the portfolio with the minimum variance. To begin, we should notice that the guarantee offered to the beneficiary who chooses a CPPI strategy with an NPV floor is lower than in the preceding situation. This occurs because, recalling the formulas (3.7) and (3.6), when the variability decreases, the value of lambda grows, and as a result, Λ_t and $B(0)$ decrease. For the values of m studied, the method with deterministic floor appears to perform worse than the one with random floor. Because, on average, the values of the final wealth increase in the random floor, while the risk measures remain zero. When we compare the outcomes acquired in this context to the prior one, we observe that the returns are substantially lower but with zero risk measures, as one could anticipate. This may need the employment of considerably higher multiplier values in order to increase returns while keeping risk measures under control.
To determine the best strategy for each type of member, we can consider a variety of factors, including the multiplier, the percentage of contributions that go to guarantee, the percentage of income that you want to contribute, and, of course, the riskiness and performance levels of the risky sector.

We now propose potential adjustments that might be the focus of future research. In the case of a market crash, the danger that portfolio assets will be totally invested in the risk-free asset with no chance of recovery is increased. This issue may be avoided by adjusting the floor in response to market conditions. Balder S.,Mahayni A. (2010) [4] provide a comprehensive examination of the topic in discrete-time commerce. They examine a generalized version of a cash-lock by focusing on the probability that the investment quote recovers from small values. Even though the dynamic versions of option-based techniques and proportional portfolio insurance methods have the same expected return, they exhibit significantly distinct cash-lock behavior. Another issue emerges in the event of a sudden gain in the market, when the minimum value (the bond floor) becomes negligible in comparison to the portfolio's value. As this may result in a huge potential loss and prevent the

advantage of the expanded market, a ratchet mechanism in the floor might be a suitable solution. Under-discrete time trading with a defined development strategy is proposed by Boulier J.F., Kanniganti A. (2005) [11]. Conditional floors provide an alternate option; see Ameur H.B., Prigent J.L. (2011) [1] and later (2018) [2]. According to [1], the floor is adjusted based on market movements and portfolio management objectives. These extensions enable the investor to profit from market performance while, for example, maintaining a portion of previous profits.
In [2], the authors extend the previous paper's results [1]. In this framework, they focus on both the margin and the ratchet-based methods. The first technique prevents the portfolio from being monetized, which is known as the cash-lock risk; the second method allows to maintain a part of your previous earnings regardless of potential large financial market drawdowns, which corresponds to ratchet effects. In this situation, the sequence of conditional floors is now growing, as opposed to the prior case, when it was decreasing. This technique also means that portfolio exposure be minimized in order to manage the risk of falling below the new floor. We may combine our proposed methods with the conditional floors, examined in the above mentioned papers, to produce more flexible and adaptive strategies to market developments and fund members' preferences. It is also possible to better adjust the strategy's components to reality. One might consider adopting more realistic dynamics of income from labor (see ad example Guvenen F. (2009) [22]), and the risky component of the strategy that can best be formed by a jump process (see Cont R., Tankov P., (2009)[18]).

Appendix A

Stochastic Calculus

A.1 Brownian Motion

We define Brownian motion and show its basic properties.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t,\ t \geq 0\}$ a filtration on it.
$W = \{W_t, t \geq 0\}$ or $W(t)$ is called standard Brownian Motion adapted to filtration $\{\mathcal{F}_t,\ t \geq 0\}$ if $W(t)$ is a stochastic process that has the following properties:

- $\mathbb{P}(W_0 = 0) = 1$
- The process has independent increments, i.e. $W_t - W_s$ is independent from $\mathcal{F}_s$ $\forall\ t \geq s$. In other terms:
  $\forall\ t_k$ in $0 = t_0 < t_1 < ... < t_k < ... < t_n \longrightarrow W(t_k) - W(t_{k-1})$ are independent.
- $W(t_k) - W(t_{k-1})$ is normally distributed $\forall k$ with:

$$\mathbb{E}(W(t_k) - W(t_{k-1})) = 0$$

$$\mathbb{V}(W(t_k) - W(t_{k-1})) = |t_k - t_{k-1}|$$

  i.e. $W(t_k) - W(t_{k-1}) \sim N(0, |t_k - t_{k-1}|)$.

Taking up the definition offered by [35], we better clarify what we mean with filtration of a Brownian motion.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian Motion W_t, $t \geq 0$. A filtration for a Brownian Motion is a collection of $\sigma - algebras \mathcal{F}_t$, $t \geq 0$, satisfying:

1. **Information accumulates** For $0 \geq s < t$, every set in $\mathcal{F}_s$ is also in $\mathcal{F}_t$. In other words, there is at least as much information available at the later time $\mathcal{F}_t$ as there is at the earlier time $\mathcal{F}_t$.
2. **Adaptivity** For each $t \geq 0$, the Brownian Motion at time t is $\mathcal{F}_t$ -measurable. In other words, the information available at time t is sufficient to evaluate the Brownian Motion W_t at that time.

3. **Independence of future increments** For $0 \geq t < u$, the increment $W_u - W_t$ is independent of $\mathcal{F}_t$. In other words, any increment of the Brownian motion after time t is independent of the information available at time t.

   Let X_t, $t \geq 0$, be a stochastic process. We say that X_t is adapted to the filtration $\mathcal{F}_t$ if for each $t \geq 0$ the random variable X_t is $\mathcal{F}_t$ -measurable.

Theorem The Brownian motion is a Martingale. It satisfies the three properties of a martingale, that is:

- W_t is $\mathcal{F}_t$ -measurable for each $t \geq 0$
- $\mathbb{E}(|W_t|) < \infty$ for $t \geq 0$
- $\mathbb{E}(W_t|\mathcal{F}_s) = W_s \qquad \forall\, t > s$

$$\mathbb{E}(W_t|\mathcal{F}_s) = \mathbb{E}(W_t - W_s + W_s|\mathcal{F}_s) = \mathbb{E}(W_t - W_s|\mathcal{F}_s) + W_s = W_s \qquad \forall\, t > s \tag{A.1}$$

Theorem The Brownian motion is a Markovian process. A process with independent increments is always Markovian:

$$\mathbb{P}(W_t \in A|\mathcal{F}_s) = \mathbb{P}(W_t \in A|W_s)$$

$\forall\, t \geq s$ and for each Borel set A. The property is stated as follows: the development of the Brownian motion after time s is controlled only by the value of W at time s. In assessing probability to the first member, no $\mathcal{F}_s$ events other than those of the type W_s play a role, and this is summed up by saying that W has a *"Markovian memory"*. Another approach to define and prove Markov's property is as follows: given $z < s < t$, the r.v. W_z and W_t are independent conditional on W_s. This intuitively corresponds with the notion that the past (W_z) and future (W_t) are independent, known the present (W_s).

We define the transition density for Brownian motion:

$$P(W_t \in dy|W_s = x) = \frac{e^{\frac{-(y-x)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}dy \tag{A.2}$$

Conditioned on the information in $\mathcal{F}_s$ (which contains all the information obtained by observing the Brownian motion up to and including time s), the conditional density of W_t is (A.2). This density is normal with mean W_s and variance $t - s$. In particular, the only information from $\mathcal{F}_s$ that is relevant is the value of W_s. The fact that only W_s is relevant is the essence of the Markov property.

We present the most significant Brownian motion properties.
The trajectories of the Brownian motion are:

- continuous
- not differentiable

The first item could be proof from a Kolmogorov theorem:
Kolmogorov Theorem Given the process X(t), $t \geqslant 0$ and be given $g = g(h)$ and $q = q(h)$ two non-decreasing monotone even functions for $h > 0$, and $\sum_{n=1}^{\infty} g(2^{-n}) < \infty$, $\sum_{n=1}^{\infty} 2^n q(2^{-n}) < \infty$;
if $\forall t, t+h \in (a,b)$,

$$P(|X(t+h) - X(t)| > g(h)) < q(h)$$

$X(t)$ ha continuous trajectories.

Furthermore, a well-known result due to Lévy shows that the quadratic variation of the Brownian motion in the time interval $[0,t]$ is equal to t. This result is related to the fact that the variation before motion Brownian in $[0,t]$ is infinite. Obviously this latter property derives from the non-differentiability of its trajectories:
For the Brownian W_t:

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} [W(kt2^{-n}) - W((k-1)t2^{-n})]^2 = t \tag{A.3}$$

The demonstrations of the posted properties may be found in detail in [5].

A.2 Itô's Formula

A fundamental result of the stochastic calculus is the Itô's formula. Considering a function $y = f(x,t)$ with $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$, with partial derivatives continuous and limited. Then for a process $Y(t) = f(W(t),t)$ the Itô's formula gives a representation of dY as follows:

$$dY = \frac{\partial f}{\partial x}(W(t),t)dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(W(t),t)dt + \frac{\partial f}{\partial t}(W(t),t)dt \tag{A.4}$$

(A.4) differs to normal calculus for the term with the second order derivative. The explanation for this outcome may be observed by writing the Y process increment and extending the f function as shown below.

$$\begin{aligned} Y(t+\Delta t) - Y(t) &= f(W(t+\Delta t), t+\Delta t) - f(W(t),t) \\ &= f(W(t) + \Delta W, t+\Delta t) - f(W(t),t) \\ &= \frac{\partial f}{\partial x}(W(t),t)\Delta W + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(W(t),t)\Delta^2 W + \frac{\partial f}{\partial t}(W(t),t)\Delta t + o(\Delta t) \end{aligned}$$

The associate dt term descends from the fact that $\Delta^2 Z \simeq \Delta t$.

Supposed that $\frac{\partial f}{\partial x}(W(s), s) \in \mathcal{H}_2[0, t]$ for $s \in (0, t)$. Now we'll define Itô's formula with the following relation:

$$\begin{aligned} & f(W(t), t) - f(W(0), 0) \\ & = Y(t) - Y(0) \\ & = \int_0^t \frac{\partial f}{\partial x}(W(s), s) dW(s) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(W(s), s) ds + \int_0^t \frac{\partial f}{\partial t}(W(s), s) ds \end{aligned} \tag{A.5}$$

We invite you to consult [5] for a demonstration and in-depth discussion of stochastic integration.

A.3 Fenton-Wilkinson Approximation Method

Fenton and Wilkinson [20] estimate the probability density function of a sum of lognormal random variables again using a log-normal distribution with the same mean and variance. The Fenton-Wilkinson approximation (also known as the FW technique) assumes that the sum of independent log-normal random variables has a log-normal distribution.
Let $X_1, X_2, \dots, X_n$ be n independent log-normally distributed random variables, i.e. $X_i \sim LogN(\mu_i; \sigma_i)$ for $i = 1, 2, \dots, n$. Then, each X_i can be written as $X_i = e^{Y_i}$ where $Y_i \sim N(\mu_i; \sigma_i)$. The general closed form expressions of the probability densitiy function and the cumulative densitiy function of the sum $Z = \sum_{i=1}^{n} X_i$ are not available. However, FW method suggests that this sum can be approximated by a new log-normal random variable Z, and the new distribution can be specified by matching the moments of Z and the moments of the sum.

We now show the approximation for $n = 2$, i.e. for the sum of two log-normal random variables. Consider the summation $aX_1 + bX_2$ where $X_i \sim LogN(\mu_i; \sigma_i)$ for $i = 1, 2$ and a, b are real constants. By FW method, we approximate this sum with a log-normal random variable $Z = e^Y$ with $Y \sim N(\mu; \sigma)$. It is noted that the first and second moments of Z, as defined, are given by

$$\mathbb{E}(Z) = e^{\mu + \frac{1}{2}\sigma^2}$$

$$\mathbb{E}(Z^2) = e^{2\mu + 2\sigma^2}$$

On the other hand, the moments of $aX_1 + bX_2$ are

$$\mathbb{E}(aX_1 + bX_2) = a\mu_1 + b\mu_2$$

$$\mathbb{E}((aX_1 + bX_2)^2) = a^2\mathbb{E}(X_1^2) + b^2\mathbb{E}(X_2^2) + 2ab\mu_1\mu_2$$

Matching the moments, it leads to

$$\mu = \ln\left(\frac{(a\mu_1 + b\mu_2)^2}{(a^2\mathbb{E}(X_1^2) + b^2\mathbb{E}(X_2^2) + 2ab\mu_1\mu_2)^{\frac{1}{2}}}\right) \tag{A.6}$$

$$\sigma^2 = \ln\left(\frac{a^2\mathbb{E}(X_1^2) + b^2\mathbb{E}(X_2^2) + 2ab\mu_1\mu_2}{((a\mu_1 + b\mu_2)^2)}\right) \tag{A.7}$$

Appendix B

Risk-Neutral Pricing

Risk-Neutral pricing is a method extensively used in quantitative finance to generate derivative prices. It is a powerful method for computing prices of derivative securities, but it is fully justified only when it is accompanied by a hedge for a short position in the security being priced. [35]

B.1 Girsanov Theorem

Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a non negative random variable Z satisfying $\mathbb{E}(Z) = 1$. We defined a new probability measure $\mathbb{Q}$ by the formula:

$$\mathbb{Q}(A) = \int_A Z(\omega) dP(\omega), \qquad \forall\ A \in \mathcal{F} \tag{B.1}$$

Any random variable X has now two expectations, one under the original measure $\mathbb{P}$ and one under $\mathbb{Q}$. The relations between these two measures is given by:

$$\mathbb{E}^{\mathbb{Q}}(X) = \mathbb{E}^{\mathbb{P}}(XZ) \tag{B.2}$$

If $\mathbb{P}(Z > 0) = 1$, then $\mathbb{P}$ and $\mathbb{Q}$ agree which sets have probability zero and so (B.2) could be written as:

$$\mathbb{E}^{\mathbb{P}}(X) = \mathbb{E}^{\mathbb{Q}}\left(\frac{X}{Z}\right) \tag{B.3}$$

Z is the Radon-Nikodym derivative of $\mathbb{Q}$ respect to $\mathbb{P}$, and:

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{B.4}$$

In the case of a finite probability model, we actually have

$$Z = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)} \tag{B.5}$$

If we multiply both sides of (B.5) by $\mathbb{Q}(\omega)$ and then sum over ω in a set A, we obtain

$$\mathbb{Q}(A) = \sum_{\omega \in A} Z(\omega) P(\omega), \qquad \forall\ A \in \Omega \tag{B.6}$$

Lemma. Be t in $[0, T]$ and let Y be a $\mathcal{F}_t - measurable$ random variable. Then

$$\mathbb{E}^{\mathbb{Q}}(Y) = \mathbb{E}^{\mathbb{P}}(YZ(t)) \tag{B.7}$$

Lemma. Be t and s in $0 \leqslant s \leqslant t \leqslant T$ and let Y be a $\mathcal{F}_t - measurable$ r.v.. Then

$$\mathbb{E}^{\mathbb{Q}}[Y|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}^{\mathbb{P}}[YZ(t)|\mathcal{F}_s] \tag{B.8}$$

Girsanov Theorem - one dimension. Be W_t, for $0 \leqslant t \leqslant T$ a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t$ a filtration for the Brownian motion. Let B_t for $0 \leqslant t \leqslant T$ an adapted process. Define:

$$\rho_t = exp(-\int_0^t B_u dW_u - \frac{1}{2}\int_0^t B_u^2 du) \tag{B.9}$$

$$\tilde{W_t} = W_t + \int_0^t B_u du \tag{B.10}$$

Is also assumed that

$$\mathbb{E}\Big(\int_0^T B_u^2 \rho_u^2 du\Big) < \infty \tag{B.11}$$

Set $\rho = \rho(T)$. Then $\mathbb{E}[\rho]$=1 and under the probability measure $\mathbb{Q}$ given by (B.1), $\tilde{Z_t},\ 0 \leq t \leq T$ is a Brownian motion.

First fundamental theorem of asset pricing If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

Second fundamental theorem of asset pricing Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.

In this work is setted up an asset price model in which $\mathbb{P}$ is the actual probability measure and $\mathbb{Q}$ is the risk-neutral measure. Considering a payoff Y_T not exposed to interest rate risk. If the market is perfect complete and without arbitrages, then the value in 0 of Y_T is:

$$\mathcal{V}(0, Y_T) = \frac{\mathbb{E}_0^{\mathbb{Q}}[Y_T]}{[1 + i(0, T)]^T} \tag{B.12}$$

The risk neutral expectation is the market equivalent, i.e. the risk neutral probability measure contains the risk premium and so is a risk-adjusted measure. Based on what stated before in Girsanov theorem, $\mathbb{Q}$ is also known as equivalent martingale measure, specified in correspondence of a numeraire $\mathcal{N}$, that is the unit of measurement of the measure, $\exists$ a one-to-one correspondence between the couple (Measure, Numeraire), for the measure $\mathbb{Q}$ the numeraire is the money market account δ:

$$\delta_T = e^{\int_0^T r(u)du} \tag{B.13}$$

r spot rate. So the money market account is the payoff in T of an investment in $t = 0$ of a one unit of currency in a roll-over strategy in ZCB with infinitesimal maturity. If the market is complete perfect and free from arbitrages than $(\mathbb{Q}, \delta)$ exists and is unique.

Appendix C

EIOPA Term Structures

EIOPA has decided to publish the relevant risk-free interest rates term structure for integer maturities from one year to 150 years. The interpolation, where necessary, and extrapolation of interest rates have been developed applying the Smith-Wilson method. This method is of course not the only one possible method for the extrapolation of interest rates. All methods have their pros and cons. The Smith-Wilson method has been applied during the last years of the development of the Solvency II framework, and in particular in the fifth Quantitative Impact Study (QIS5) and in the Long-term Guarantees Assessment (LTGA) that has underpinned the political agreement of the Omnibus II Directive. EIOPA will however carefully monitor market developments, and their influence on the implementation of the Smith-Wilson method. For each currency the basic risk-free interest rate term structure is constructed from risk-free interest rates for a finite number of maturities. Both the interpolation between these maturities, where necessary, and the extrapolation beyond the last liquid point (LLP)[1] are based on the Smith-Wilson methodology. The convergence point is the maximum of (LLP+40) and 60 years. Consequently, the convergence period is the maximum of (60-LLP) and 40 years. [44]

Let us reconstruct the steps of the Smith and Wilson model.

Interpolation

The Smith-Wilson method takes care that the present value function of the derived term structure exactly agrees with the empirical data for the observable maturities.
We are looking for a function that passes through n points. The points are identified in the following manner:

$$(x_i, y_i), \qquad \forall\, i = 1, \dots, n$$

As a result, we're searching for a function with as many arguments as the n points under consideration.
It are used *Lagrange polynomial* in such a way that polynomials of n-1 degrees are detected that are worth 1 in the point we need to interpolate and 0 otherwise.

[1]For EUR the LLP is equal to 20.

- for $n = 2$ passes a straight line:

$$\begin{aligned} L_1(x) =& \frac{x - x_2}{x_1 - x_2} \longmapsto \begin{cases} if \quad x = x_1 : L_1(x_1) = 1 \\ if \quad x = x_2 : L_1(x_2) = 0 \end{cases} \\ or & \\ L_2(x) =& \frac{x - x_1}{x_2 - x_1} \longmapsto \begin{cases} if \quad x = x_1 : L_2(x_1) = 0 \\ if \quad x = x_2 : L_2(x_2) = 1 \end{cases} \end{aligned} \tag{C.1}$$

The line that passes through two points is
$f(x) = y_1 L_1(x) + y_2 L_2(x)$

for n=3:

$$\begin{aligned} L_1(x) =& \frac{x - x_2}{x_1 - x_2}\frac{x - x_3}{x_1 - x_3} \longmapsto \begin{cases} if \quad x = x_1 : L_1(x_1) = 1 \\ if \quad x = x_2 : L_1(x_2) = 0 \\ if \quad x = x_3 : L_1(x_3) = 0 \end{cases} \\ or & \\ L_2(x) =& \frac{x - x_1}{x_2 - x_1}\frac{x - x_3}{x_2 - x_3} \longmapsto \begin{cases} if \quad x = x_1 : L_2(x_1) = 0 \\ if \quad x = x_2 : L_2(x_2) = 1 \\ if \quad x = x_3 : L_2(x_3) = 0 \end{cases} \\ or & \\ L_3(x) =& \frac{x - x_1}{x_3 - x_1}\frac{x - x_2}{x_3 - x_2} \longmapsto \begin{cases} if \quad x = x_1 : L_2(x_1) = 0 \\ if \quad x = x_2 : L_2(x_2) = 0 \\ if \quad x = x_3 : L_2(x_3) = 1 \end{cases} \end{aligned} \tag{C.2}$$

The parabola that passes through three points is
$f(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$

- $\forall$ n the function that passes through n points is a $n - 1$ polynomial, given by:

$$f(x) = \sum_{i=0}^{n} y_i L_i(x)$$

We must address a crucial issue: the greater the degree of polynomial, the more oscillating the function. A picewise interpolation is a reasonable solution. So instead of using an interpolating curve for every point, we partition the n points into subgroups and find an interpolating curve for each subgroup. The trade-off is

in the choice of a third degree polynomial.
With the cubic spline, is needed a point of continuity between the sections. So $n+1$ point corresponds to n intervals.
In the i-interval one has $f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$.
So 4 parameters for n unknowns (n intervals) $\Rightarrow 4n$ unknowns. So we need $4n$ conditions. Passage for $n+1$ points corresponds $n+1$ conditions. We must then ensure that the interpolation curves and their corresponding (first and second) derivations have the same value in the $(n+1)-2 = n-1$ contact points. It are added $3(n-1)$.
In total we have $3(n-1)+n+1 = 4n-2$ conditions, so are needed two more conditions. We impose that:

$$f'(x_0) = f'(x_n) = 0$$

.
The one described is the Natural Cubic Spline. The spline used by EIOPA is the Tension Spline, it is an intermediate splines between linear and cubic. The tension spline are characterized by an exogenous parameter, a tension parameter such that if it tends to 0, obtaining the cubic spline, if it tends to infinity, the linear spline is obtained.
Consider the reference instruments are zero coupon government bonds[2]. EIOPA's Smith and Wilson model interpolate on discount factor $v(t, t+\tau)$ and is given by:

$$v(t, t+\tau) = e^{-h_\infty \tau} + \sum_{j=1}^{N} \xi_j W(\tau, \tau_j) \tag{C.3}$$

for a given τ and s, the $W(\tau, s$ is such that:

$$W(\tau, s) = e^{-h_\infty(\tau+s)}[\alpha min(\tau, s) - e^{-\alpha max(\tau,s)} sinh(\alpha min(\tau, s)] \tag{C.4}$$

the parameters are $-h_\infty$, α exogenous, and ξ_i, τ_i given by the market for i=1,...,N. h_∞ is such that $h_\infty = \lim_{\tau\to\infty} h^{SW}(t, t+\tau)$.
Given $v^{SW}(t, t+\tau)$ is possible to write $h^{SW}(t, t+\tau) = -\frac{1}{\tau} lnv(t, t+\tau)$, thus:

$$\lim_{\tau\to\infty} -\frac{1}{\tau} \ln\left(v(t, t+\tau)\right) = h_\infty \tag{C.5}$$

The parameter α that controls the convergence speed (how quickly the curve reaches the asymptotic value as τ increases) is set at the lowest value that produces a term structure reaching the convergence tolerance[3] of the ultimate forward rate (UFR) by the convergence point.

A particular maturity τ^* is established in order to choose the value of α, for which

$$|h(t, t+\tau^*) - h_\infty| < \varepsilon$$

[2] the market interest rates to be used as inputs are the zero coupon rates after deduction of the credit and currency risk adjustments.
[3] The convergence tolerance is set at 1 bp and a lower bound for alpha is set at 0.05.

EIOPA does not want the asymptote to be reached in a very high number of years, therefore it imposes an α such that the structure arrives at the asymptote in generally a $\tau^* = 60$ (convergence point).
The other parameters are given from market. We write equations in matrix form:

$$\begin{bmatrix} v(t, t+\tau_1) \\ v(t, t+\tau_2) \\ ... \\ v(t, t+\tau_N) \end{bmatrix} = \begin{bmatrix} e^{-h_\infty \tau_1} \\ e^{-h_\infty \tau_2} \\ ... \\ e^{-h_\infty \tau_N} \end{bmatrix} + \begin{bmatrix} W(\tau_1, \tau_1) + ... + W(\tau_1, \tau_N) \\ W(\tau_2, \tau_1) + ... + W(\tau_2, \tau_N) \\ ... \\ W(\tau_N, \tau_1) + ... + W(\tau_N, \tau_N) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ ... \\ \xi_N \end{bmatrix} \tag{C.6}$$

or simply:

$$\vec{v} = \vec{\eta} + \mathbf{W}\vec{\xi} \tag{C.7}$$

$\vec{v}$, $\vec{\eta}$ *and* $\mathbf{W}$ are known, $\vec{\xi}$ is the unidentified vector. Thus, by solving the following linear equation,

$$\vec{\xi} = \mathbf{W}^{-1}(\vec{v} - \vec{\eta}) \tag{C.8}$$

we obtain the desired numbers.

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